

UNIT-1

Vector Differentiation & Integration

Vector differentiation

Vector :- A physical quantity which has both magnitude and direction is called a vector.

Ex:- velocity, acceleration, force ... etc

Scalar :- A physical quantity which has only magnitude is called a scalar.

Ex:- Mass, length, area, volume ... etc.

Unit vector :- A vector of length one unit is called a unit vector.

If $\vec{a}$ is any non zero vector, then $\frac{\vec{a}}{|\vec{a}|}$ is a unit vector in the direction of $\vec{a}$. It is denoted by " $\hat{a}$ ".

Position vector :- Let 'O' be the fixed point in the space called as origin. If 'P' is any point in the space then $\vec{OP}$ is called position vector of 'P' with respect to O.

If A & B are two points in the space then

$$\vec{AB} = \vec{OB} - \vec{OA} \text{ by triangle law.}$$

If $\vec{r}$ is the position vector of the point $P(x, y, z)$ then, $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

Note :- * If $\vec{a} = x_1\vec{i} + y_1\vec{j} + z_1\vec{k}$ & $\vec{b} = x_2\vec{i} + y_2\vec{j} + z_2\vec{k}$ be two vector then $\vec{a} \cdot \vec{b} = (x_1\vec{i} + y_1\vec{j} + z_1\vec{k}) \cdot (x_2\vec{i} + y_2\vec{j} + z_2\vec{k})$
 $= x_1x_2 + y_1y_2 + z_1z_2$

$$* \vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \begin{matrix} \vec{i} \times \vec{j} = \vec{k} \\ \vec{j} \times \vec{k} = \vec{i} \\ \vec{k} \times \vec{i} = \vec{j} \end{matrix}$$

vector differentiation :-

Let $f(t) = f_1(t)\bar{i} + f_2(t)\bar{j} + f_3(t)\bar{k}$ is a vector point function which varies with respect to 't'. Then derivative of $f(t)$ with respect to 't' is defined as

$$\frac{d}{dt} f(t) = \frac{d}{dt} (f_1(t)\bar{i} + f_2(t)\bar{j} + f_3(t)\bar{k})$$

$$\frac{dF}{dt} = f_1'(t)\bar{i} + f_2'(t)\bar{j} + f_3'(t)\bar{k}$$

Similarly

$$\frac{d^2 F}{dt^2} = f_1''(t)\bar{i} + f_2''(t)\bar{j} + f_3''(t)\bar{k}$$

Scalar & vector point functions :-

Consider a region in two or 3D space. To each point $P(x, y, z)$, we associate a unique ^{real} number ϕ . This ' ϕ ' is called a scalar point function. On the region.

Similarly, if to each point $P(x, y, z)$ we associate a unit vector $\bar{f}(x, y, z)$. $\bar{f}$ is called a vector point function.

Ex :- * Take a heated solid. At each point $P(x, y, z)$ of the solid, there will be temperature ' T '. This ' T ' is called scalar point function.

* Consider a particle moving in space at each point ' P ' and its path, the particle will be having a velocity ($\bar{v}$) which is a vector point function.

Similarly acceleration of the particle a is also a vector point function.

Vector differential operator :-

It is denoted by $\text{del } (\nabla)$ and is defined as

$$\nabla = \bar{i} \frac{\partial}{\partial x} + \bar{j} \frac{\partial}{\partial y} + \bar{k} \frac{\partial}{\partial z}$$

Gradient of scalar point function :-

Let $\phi(x, y, z)$ be a scalar point function. The gradient of ϕ is denoted by "grad ϕ " or " $\nabla \phi$ " and is defined by

$$\nabla \phi = \bar{i} \frac{\partial \phi}{\partial x} + \bar{j} \frac{\partial \phi}{\partial y} + \bar{k} \frac{\partial \phi}{\partial z}$$

Properties :-

* If f & g are two scalar point functions then

$$\text{grad}(f \pm g) = \text{Grad } f + \text{Grad } g$$

$$* \text{grad}(fg) = f \text{ grad } g + g \text{ grad } f$$

$$* \text{grad}\left(\frac{f}{g}\right) = \frac{g \text{ grad } f - f \text{ grad } g}{g^2} \quad (g \neq 0)$$

* If f is a scalar point function & 'c' is a constant then $\text{grad}(cf) = c \text{ grad } f$.

Tangent vector to a curve :-

Let $\bar{r} = f(t)$ be a vector curve then $\frac{d\bar{r}}{dt}$ is the tangent to the curve. Where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and x, y, z are functions of 't'. Then

$$\frac{d\bar{r}}{dt} = \frac{d}{dt} x\bar{i} + \frac{d}{dt} y\bar{j} + \frac{d}{dt} z\bar{k}$$

The unit tangent vector is $\frac{\frac{d\bar{r}}{dt}}{\left|\frac{d\bar{r}}{dt}\right|}$ Where $\frac{d\bar{r}}{dt}$

represent a vector along tangent to the curve

Normal vector to the surface :-

Let $\phi(x, y, z) = k$ be the surface then the normal vector to the surface is equal to "grad ϕ "

The unit normal vector to the surface is $\frac{\nabla \phi}{|\nabla \phi|}$

Angle between two surfaces:-

Let $f(x, y, z) = k$ and $\phi(x, y, z) = c$ be two surfaces.

Then the angle between two surfaces at a point 'p' is the angle between the normals to the given surfaces at that point.

Let ' θ ' be the angle between two normal vector then

$$\cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|}$$

Directional derivative :-

Let $\phi(x, y, z)$ is a scalar point function then the directional derivative of ϕ at a point 'p' in the direction of $\vec{n}$ is given defined by

$$D.D = [\text{grad } \phi]_p \cdot \frac{\vec{n}}{|\vec{n}|} \quad \therefore \text{where } \hat{e} = \frac{\vec{n}}{|\vec{n}|}$$

Where $\hat{e}$ is the unit vector in the direction of $\vec{n}$.

$$D.D = [\nabla \phi]_p \cdot \hat{e}$$

Note :-

* Greatest value of directional derivative of ϕ at a point p is equal to $|\text{grad } \phi|$

Problems :-

1. Find grad f where (a) $f = x^3 + y^3 + 3xyz$. (b) $f = x^2y + y^2x + z^2$ at the point $(1, 1, -2)$

Sol:- (b) Given $f = x^2y + y^2x + z^2 \rightarrow \textcircled{1}$

we know that

$$\text{grad } f = \nabla f = \hat{i} \frac{\partial f}{\partial x} + \hat{j} \frac{\partial f}{\partial y} + \hat{k} \frac{\partial f}{\partial z}$$

diff eqn $\textcircled{1}$, partially with respect to x, y & z we get

$$\frac{\partial f}{\partial x} = 2xy + y^2$$

$$\frac{\partial f}{\partial z} = 2z$$

$$\frac{\partial f}{\partial y} = x^2 + 2yx$$

$$\text{grad } f = \bar{i}(2xy+y^2) + \bar{j}(x^2+2xy) + \bar{k}(2z)$$

Given point $(1, 1, 2)$

$$\begin{aligned}\text{grad } f &= [2(1)(1)+1] \bar{i} + (1+2(1)(1)) \bar{j} + 2(2) \bar{k} \\ &= 3\bar{i} + 3\bar{j} + 4\bar{k}\end{aligned}$$

(a) Given $f = x^3 + y^3 + 3xyz$

$$\frac{\partial f}{\partial x} = 3x^2 + 3yz$$

$$\frac{\partial f}{\partial z} = 3xy$$

$$\frac{\partial f}{\partial y} = 3y^2 + 3xz$$

$$\text{grad } f = (3x^2 + 3yz) \bar{i} + (3y^2 + 3xz) \bar{j} + 3xy \bar{k}$$

2. If $\phi = 2xz^4 - x^2y$, find $|\nabla\phi|$ at the point $(2, -2, -1)$

Sol:- Given

$$\phi = 2xz^4 - x^2y$$

$$\nabla\phi = \bar{i} \frac{\partial\phi}{\partial x} + \bar{j} \frac{\partial\phi}{\partial y} + \bar{k} \frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = 2z^4 - 2xy$$

$$\frac{\partial\phi}{\partial z} = 8xz^3$$

$$\frac{\partial\phi}{\partial y} = -x^2$$

$$\text{Now, } \nabla\phi = (2z^4 - 2xy) \bar{i} - x^2 \bar{j} + 8xz^3 \bar{k}$$

Given point $(2, -2, -1)$

$$\nabla\phi = [2(-1)^4 - 2(2)(-2)] \bar{i} - 2^2 \bar{j} + 8(2)(-1)^3 \bar{k}$$

$$= (2+8) \bar{i} - 4\bar{j} - 16\bar{k}$$

$$\nabla\phi = 10\bar{i} - 4\bar{j} - 16\bar{k}$$

$$\therefore |\nabla\phi| = \sqrt{(10)^2 + (-4)^2 + (-16)^2}$$

$$= \sqrt{100 + 16 + 256}$$

$$= \sqrt{372}$$

3. Find the unit normal vector to the surface

(i) $x^3 + y^3 + 3xyz = 3$ at $(1, 2, -1)$

(ii) $xy + yz + zx$ at $(1, 1, 1)$

(iii) $x^2 + y^2 + z^2 = 26$ at $(2, 2, 3)$

Sol:- (iii) let $\phi = x^2 + y^2 + 2z^2 - 26$

We know that normal vector to the surface ϕ is equal to $\text{grad } \phi$

$$\text{grad } \phi = \nabla \phi = \bar{i} \frac{\partial \phi}{\partial x} + \bar{j} \frac{\partial \phi}{\partial y} + \bar{k} \frac{\partial \phi}{\partial z}$$

$$\frac{\partial \phi}{\partial x} = 2x \quad \frac{\partial \phi}{\partial y} = 2y \quad , \quad \frac{\partial \phi}{\partial z} = 4z$$

$$\text{grad } \phi = \nabla \phi = 2x \bar{i} + 2y \bar{j} + 4z \bar{k} = \bar{n} \text{ is normal vector to the surface } \phi$$

then $\nabla \phi$ at $(2, 2, 3)$

$$\begin{aligned} \nabla \phi &= 2(2) \bar{i} + 2(2) \bar{j} + 4(3) \bar{k} \\ &= 4\bar{i} + 4\bar{j} + 12\bar{k} = \bar{n} \end{aligned}$$

$\nabla \phi$

$\therefore$ unit normal vector to the surface

$$= \frac{\bar{n}}{|\bar{n}|}$$

$$= \frac{4\bar{i} + 4\bar{j} + 12\bar{k}}{\sqrt{4^2 + 4^2 + 12^2}}$$

$$= \frac{4\bar{i} + 4\bar{j} + 12\bar{k}}{\sqrt{16 + 16 + 144}}$$

$$= \frac{4(\bar{i} + \bar{j} + 3\bar{k})}{\sqrt{176}}$$

$$= \frac{4(\bar{i} + \bar{j} + 3\bar{k})}{4\sqrt{11}}$$

$$= \frac{\bar{i} + \bar{j} + 3\bar{k}}{\sqrt{11}}$$

(i) $\phi = x^3 + y^3 + 3xyz = 3$ at $(1, 2, -1)$

$$\text{grad } \phi = \nabla \phi = \bar{i} \frac{\partial \phi}{\partial x} + \bar{j} \frac{\partial \phi}{\partial y} + \bar{k} \frac{\partial \phi}{\partial z}$$

$$\frac{\partial \phi}{\partial x} = 3x^2 + 3yz$$

$$\frac{\partial \phi}{\partial z} = 3xy$$

$$\frac{\partial \phi}{\partial y} = 3y^2 + 3xz$$

$$\text{grad } \phi = (3x^2 + 3yz) \bar{i} + (3y^2 + 3xz) \bar{j} + 3xy \bar{k}$$

at $(1, 2, -1)$

$$\begin{aligned}\text{grad } \phi &= [3(1)^2 + 3(2)(-1)]\bar{i} + [3(2)^2 + 3(1)(-1)]\bar{j} + 3(1)(2)\bar{k} \\ &= (3-6)\bar{i} + (12-3)\bar{j} + 6\bar{k} \\ &= -3\bar{i} + 9\bar{j} + 6\bar{k}\end{aligned}$$

Unit normal vector to the surface

$$\begin{aligned}\bar{n} &= \frac{-3\bar{i} + 9\bar{j} + 6\bar{k}}{\sqrt{(-3)^2 + (9)^2 + 6^2}} \\ &= \frac{-3(\bar{i} - 3\bar{j} - 2\bar{k})}{\sqrt{9 + 81 + 36}} \\ &= \frac{-3(\bar{i} - 3\bar{j} - 2\bar{k})}{\sqrt{126}} \\ &= \frac{-3(\bar{i} - 3\bar{j} - 2\bar{k})}{3\sqrt{14}} = \frac{-(\bar{i} - 3\bar{j} - 2\bar{k})}{\sqrt{14}}\end{aligned}$$

(ii) Given $\phi = xy + yz + zx$ at $(1, 1, 1)$

$$\begin{aligned}\text{grad } \phi &= \bar{i} \frac{\partial \phi}{\partial x} + \bar{j} \frac{\partial \phi}{\partial y} + \bar{k} \frac{\partial \phi}{\partial z} \\ \frac{\partial \phi}{\partial x} &= y+z, \quad \frac{\partial \phi}{\partial y} = x+z, \quad \frac{\partial \phi}{\partial z} = y+x\end{aligned}$$

at $(1, 1, 1)$

$$\text{grad } \phi = (y+z)\bar{i} + (x+z)\bar{j} + (y+x)\bar{k}$$

at $(1, 1, 1)$

$$\begin{aligned}\text{grad } \phi &= (1+1)\bar{i} + (1+1)\bar{j} + (1+1)\bar{k} \\ &= 2\bar{i} + 2\bar{j} + 2\bar{k} \\ &= 2(\bar{i} + \bar{j} + \bar{k})\end{aligned}$$

$\therefore$ Unit normal vector to the surface $= \frac{\bar{n}}{|\bar{n}|}$

$$= \frac{2(\bar{i} + \bar{j} + \bar{k})}{\sqrt{4+4+4}}$$

$$= \frac{2(\bar{i} + \bar{j} + \bar{k})}{\sqrt{12}}$$

$$= \frac{2(\bar{i} + \bar{j} + \bar{k})}{2\sqrt{3}}$$

$$= \frac{\bar{i} + \bar{j} + \bar{k}}{\sqrt{3}}$$

48. Find the angle between the surfaces (i) $xy^2z = 3x + z^2$ and $3x^2 - y^2 + 2z = 1$ at $(1, -2, 1)$.

(ii) $x^2 + y^2 + z^2 = 9$, $x^2 + y^2 - z = 3$ at $(2, -1, 2)$

(iii) $x^2 + y^2 + z^2 = 29$, $x^2 + y^2 + z^2 + ux - 6y - 8z - u7 = 0$ at $(4, -3, 2)$

Sol: (iii) let $\phi = x^2 + y^2 + z^2 - 29$

$f = x^2 + y^2 + z^2 + ux - 6y - 8z - u7$ be the given surfaces

we know that the angle between the two surfaces at a point (x_1, y_1, z_1) is the angle between two normals to the surfaces at that point

$$\text{Now } \nabla\phi = \bar{i} \frac{\partial\phi}{\partial x} + \bar{j} \frac{\partial\phi}{\partial y} + \bar{k} \frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = 2x, \quad \frac{\partial\phi}{\partial y} = 2y, \quad \frac{\partial\phi}{\partial z} = 2z$$

$$\nabla\phi = 2x\bar{i} + 2y\bar{j} + 2z\bar{k}$$

at $(4, -3, 2)$

$$\begin{aligned} \nabla\phi &= 2(4)\bar{i} + 2(-3)\bar{j} + 2(2)\bar{k} \\ &= 8\bar{i} - 6\bar{j} + 4\bar{k} = \bar{n}_1 \text{ is the normal vector to the surface } \phi \end{aligned}$$

let $f = x^2 + y^2 + z^2 + ux - 6y - 8z - u7$

$$\nabla f = \bar{i} \frac{\partial f}{\partial x} + \bar{j} \frac{\partial f}{\partial y} + \bar{k} \frac{\partial f}{\partial z}$$

$$\frac{\partial f}{\partial x} = 2x + u, \quad \frac{\partial f}{\partial y} = 2y - 6, \quad \frac{\partial f}{\partial z} = 2z - 8$$

$$\nabla f = (2x + u)\bar{i} + (2y - 6)\bar{j} + (2z - 8)\bar{k}$$

at $(4, -3, 2)$

$$\begin{aligned} \nabla f &= (2(4) + u)\bar{i} + (2(-3) - 6)\bar{j} + (2(2) - 8)\bar{k} \\ &= 12\bar{i} - 12\bar{j} - 4\bar{k} = \bar{n}_2 \text{ is the normal vector to the surface 'f'}. \end{aligned}$$

let ' θ ' be the angle b/w the two normal vector $\bar{n}_1, \bar{n}_2$

$$\cos\theta = \frac{\bar{n}_1 \cdot \bar{n}_2}{|\bar{n}_1| |\bar{n}_2|}$$

$$\begin{aligned}\cos \theta &= \frac{(8\bar{i} - 6\bar{j} + 4\bar{k}) \cdot (12\bar{i} - 12\bar{j} - 4\bar{k})}{\sqrt{64+36+16} \sqrt{144+144+16}} \\&= \frac{96+72-16}{\sqrt{116} \sqrt{304}} \\&= \frac{152}{\sqrt{116} \cdot 4\sqrt{19}} = \frac{38}{\sqrt{116} \cdot \sqrt{19}} = \frac{38^{19}}{2\sqrt{29} \sqrt{19}} \\&= \frac{19}{\sqrt{29} \sqrt{19}} = \frac{\sqrt{19}}{\sqrt{29}}\end{aligned}$$

$$\cos \theta = \sqrt{\frac{19}{29}}$$

$$\theta = \cos^{-1} \sqrt{\frac{19}{29}} //$$

(i) $xy^2z = 3x + z^2$ and $3x^2 - y^2 + 2z = 1$ at $(1, -2, 1)$

$$\phi = 3x + z^2 - xy^2z, \quad f = 3x^2 - y^2 + 2z - 1$$

$$\nabla \phi = \bar{i} \frac{\partial \phi}{\partial x} + \bar{j} \frac{\partial \phi}{\partial y} + \bar{k} \frac{\partial \phi}{\partial z}$$

$$\frac{\partial \phi}{\partial x} = 3 - y^2z, \quad \frac{\partial \phi}{\partial y} = -2xyz, \quad \frac{\partial \phi}{\partial z} = 2z - xy^2$$

$$\nabla \phi = (3 - y^2z)\bar{i} + (-2xyz)\bar{j} + (2z - xy^2)\bar{k}$$

at $(1, -2, 1)$

$$\nabla \phi = (3 - (-2)^2(1))\bar{i} - (2(1)(-2)(1))\bar{j} + (2(1) - (1)(-2)^2)\bar{k}$$

$$= (3 - 4)\bar{i} - 2(-4)\bar{j} + (2 + 4)\bar{k}$$

$$= -\bar{i} + 4\bar{j} + 6\bar{k} = \bar{n}_1$$

$$\text{let } \nabla f = \bar{i} \frac{\partial f}{\partial x} + \bar{j} \frac{\partial f}{\partial y} + \bar{k} \frac{\partial f}{\partial z}$$

$$\frac{\partial f}{\partial x} = 6x$$

$$\frac{\partial f}{\partial y} = -2y$$

$$\frac{\partial f}{\partial z} = 2$$

$$\nabla f = \bar{i} 6x - 2y\bar{j} + 2\bar{k}$$

at $(1, -2, 1)$

$$\nabla f = 6(1)\bar{i} - 2(-2)\bar{j} + 2\bar{k}$$

$$= 6\bar{i} + 4\bar{j} + 2\bar{k} = \bar{n}_2$$

let 'θ' be the angle b/w two normal vector $\bar{n}_1, \bar{n}_2$

$$\cos \theta = \frac{\bar{n}_1 \cdot \bar{n}_2}{|\bar{n}_1| |\bar{n}_2|}$$

$$\cos \theta = \frac{(-\vec{i} + u\vec{j} + 6\vec{k}) \cdot (6\vec{i} + u\vec{j} + 2\vec{k})}{\sqrt{(-1)^2 + u^2 + 6^2} \cdot \sqrt{6^2 + u^2 + 2^2}}$$

$$= \frac{-6 + 16 + 12}{\sqrt{53} \cdot \sqrt{56}}$$

$$= \frac{22}{\sqrt{53} \cdot 2\sqrt{14}} = \frac{11}{\sqrt{14} \cdot \sqrt{53}} //$$

$$\theta = \cos^{-1}\left(\frac{11}{\sqrt{14} \sqrt{53}}\right) //$$

(ii) Given

$$\phi = x^2 + y^2 + z^2 - 9, \quad f = x^2 + y^2 - z - 3 \quad \text{at } (2, -1, 2)$$

$$\text{Now, } \nabla \phi = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}$$

$$\frac{\partial \phi}{\partial x} = 2x, \quad \frac{\partial \phi}{\partial y} = 2y, \quad \frac{\partial \phi}{\partial z} = 2z$$

$$\nabla \phi = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

at $(2, -1, 2)$ $\nabla \phi$ is

$$\nabla \phi = 2(2)\vec{i} + 2(-1)\vec{j} + 2(2)\vec{k}$$

$$= 4\vec{i} - 2\vec{j} + 4\vec{k} = \vec{n}_1$$

$$\text{let } f = x^2 + y^2 - z - 3$$

$$\nabla f = \vec{i} \frac{\partial f}{\partial x} + \vec{j} \frac{\partial f}{\partial y} + \vec{k} \frac{\partial f}{\partial z}$$

$$\text{let } \frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 2y, \quad \frac{\partial f}{\partial z} = -1$$

$$\nabla f = 2x\vec{i} + 2y\vec{j} - \vec{k}$$

at $(2, -1, 2)$ ∇f is

$$\nabla f = 2(2)\vec{i} + 2(-1)\vec{j} - \vec{k}$$

$$= 4\vec{i} - 2\vec{j} - \vec{k} = \vec{n}_2$$

let θ be the angle b/w the two surfaces normal vectors $\vec{n}_1, \vec{n}_2$

$$\cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|}$$

$$= \frac{(4\vec{i} - 2\vec{j} + 4\vec{k}) \cdot (4\vec{i} - 2\vec{j} - \vec{k})}{\sqrt{16 + 4 + 16} \sqrt{16 + 4 + 1}}$$

$$\cos \theta = \frac{16 + 4 - 4}{\sqrt{36} \sqrt{21}}$$

$$= \frac{16}{3\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$$\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right) //$$

5. Evaluate the angle between the normals to the surface

(i) $x^2 = yz$ at the points $(1, 1, 1)$ & $(2, 4, 1)$.

(ii) $xy = z^2$ at $(1, 1, 2)$ & $(3, 3, -3)$.

Sol: (i) let $\phi = x^2 - yz$

$$\nabla \phi = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}$$

$$\frac{\partial \phi}{\partial x} = 2x \quad \frac{\partial \phi}{\partial y} = -z \quad \frac{\partial \phi}{\partial z} = -y$$

$$\nabla \phi = 2x\vec{i} - z\vec{j} - y\vec{k}$$

$$\text{Now } \nabla \phi \text{ at } (1, 1, 1) = 2(1)\vec{i} - 1\vec{j} - (1)\vec{k}$$

$$= 2\vec{i} - \vec{j} - \vec{k} = \vec{n}_1 \text{ (say)}$$

$$\nabla \phi \text{ at } (2, 4, 1) = 2(2)\vec{i} - (1)\vec{j} - (4)\vec{k}$$

$$= 4\vec{i} - \vec{j} - 4\vec{k} = \vec{n}_2 \text{ (say)}$$

Let ' θ ' be the angle between two normals then

$$\cos \theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{|\vec{n}_1| |\vec{n}_2|}$$

$$= \frac{(2\vec{i} - \vec{j} - \vec{k}) \cdot (4\vec{i} - \vec{j} - 4\vec{k})}{\sqrt{4+1+1} \sqrt{16+1+16}}$$

$$= \frac{8+1+4}{\sqrt{6} \sqrt{33}}$$

$$= \frac{13}{\sqrt{198}}$$

$$\theta = \cos^{-1}\left(\frac{13}{\sqrt{198}}\right) //$$

(ii) let $\phi = xy - z^2$

$$\frac{\partial \phi}{\partial x} = y \quad \frac{\partial \phi}{\partial y} = x \quad \frac{\partial \phi}{\partial z} = -2z$$

$$\nabla\phi = y\vec{i} + x\vec{j} - 2z\vec{k}$$

at (4, 1, 2)

$$\begin{aligned}\nabla\phi &= 1\vec{i} + 4\vec{j} - 2(2)\vec{k} \\ &= \vec{i} + 4\vec{j} - 4\vec{k} = \vec{n}_1\end{aligned}$$

at (3, 3, -3)

$$\begin{aligned}\nabla\phi &= 3\vec{i} + 3\vec{j} - 2(-3)\vec{k} \\ &= 3\vec{i} + 3\vec{j} + 6\vec{k} = \vec{n}_2\end{aligned}$$

let 'O' be the angle between two normals then,

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|}$$

$$= \frac{(\vec{i} + 4\vec{j} - 4\vec{k}) \cdot (3\vec{i} + 3\vec{j} + 6\vec{k})}{\sqrt{1+16+16} \sqrt{9+9+36}}$$

$$= \frac{3+12-24}{\sqrt{33} \sqrt{54}} = \frac{-9}{\sqrt{33} \cdot 3\sqrt{6}} = \frac{-3}{\sqrt{198}}$$

$$\theta = \cos^{-1}\left(\frac{-3}{\sqrt{198}}\right)$$

* 6. Find the constant a & b such that the surface

(i) $ax^2 - byz = (a+2)x$ is orthogonal to the surface $xyz^2 = 4$ at the point (1, -1, 2).

(ii) $3x^2 - 2y^2 - 3z^2 + 8 = 0$ and $ax^2 + y^2 = bz$ at (-1, 2, 1)

Sol:- (i) let $f = ax^2 - byz - (a+2)x$ and $g = xyz^2 - u$ be the given surfaces.

Since both the surfaces are orthogonal then their normals are also orthogonal

$$\nabla f = \vec{i} \frac{\partial f}{\partial x} + \vec{j} \frac{\partial f}{\partial y} + \vec{k} \frac{\partial f}{\partial z}$$

$$\frac{\partial f}{\partial x} = 2ax - (a+2)$$

$$\frac{\partial f}{\partial y} = -bz$$

$$\frac{\partial f}{\partial z} = -by$$

$$\nabla f = (2ax - (a+2))\vec{i} - bz\vec{j} - by\vec{k}$$

at (1, -1, 2)

$$\nabla f = (2a(1) - (a+2))\vec{i} - b(2)\vec{j} - b(-1)\vec{k}$$

$$\nabla f = (2a - a - 2)\vec{i} - 2b\vec{j} + b\vec{k}$$

$$= (a-2)\vec{i} - 2b\vec{j} + b\vec{k} = \vec{n}_1$$

Surface $g = ux^2y + z^3 - u$

$$\nabla g = \vec{i} \frac{\partial g}{\partial x} + \vec{j} \frac{\partial g}{\partial y} + \vec{k} \frac{\partial g}{\partial z}$$

$$\frac{\partial g}{\partial x} = 8xy$$

$$\frac{\partial g}{\partial y} = ux^2$$

$$\frac{\partial g}{\partial z} = 3z^2$$

$$\nabla g = 8xy\vec{i} + ux^2\vec{j} + 3z^2\vec{k}$$

at $(1, -1, 2)$

$$\nabla g = 8(1)(-1)\vec{i} + u(1)^2\vec{j} + 3(2)^2\vec{k}$$

$$= -8\vec{i} + u\vec{j} + 12\vec{k} = \vec{n}_2$$

$\therefore$ the vector are orthogonal then

$$\vec{n}_1 \cdot \vec{n}_2 = 0$$

$$((a-2)\vec{i} - 2b\vec{j} + b\vec{k}) \cdot (-8\vec{i} + u\vec{j} + 12\vec{k}) = 0$$

$$-8(a-2) - 8b + 12b = 0$$

$$-8a + 16 - 8b + 12b = 0$$

$$-8a + 4b + 16 = 0$$

$$2a - b - u = 0$$

$$2a - b = 4 \rightarrow \textcircled{1}$$

Given that both the surfaces are orthogonal at

$(1, -1, 2)$ i.e., the point $(1, -1, 2)$ lies on both the surface

put $(1, -1, 2)$ in first surface

$$\Rightarrow a(1)^2 - b(-1)(2) = (a+2)(1)$$

$$a + 2b = a + 2$$

$$2b = 2$$

$$b = 1$$

from $\textcircled{1}$

$$2a - 1 = 4$$

$$2a = 5$$

$$a = \frac{5}{2}$$

$$\therefore a = \frac{5}{2} \text{ and } b = 1$$

7. Find the directional derivative of $\phi = x^2yz + uxz^2$ at $(1, -2, -1)$ in the direction of vector $2\vec{i} - \vec{j} - 2\vec{k}$.

Sol:- Given $\phi = x^2yz + uxz^2$

$$\nabla\phi = \vec{i} \frac{\partial\phi}{\partial x} + \vec{j} \frac{\partial\phi}{\partial y} + \vec{k} \frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = 2xyz + uz^2$$

$$\frac{\partial\phi}{\partial y} = x^2z$$

$$\frac{\partial\phi}{\partial z} = x^2y + 8xz$$

$$\nabla\phi = (2xyz + uz^2)\vec{i} + x^2z\vec{j} + (x^2y + 8xz)\vec{k}$$

at $(1, -2, -1)$

$$\begin{aligned}\nabla\phi &= (2(1)(-2)(-1) + 1(-1)^2)\vec{i} + (1)^2(-1)\vec{j} + (1^2(-2) + 8(1)(-1))\vec{k} \\ &= 8\vec{i} - \vec{j} - 10\vec{k}\end{aligned}$$

Let $\hat{e}$ is the unit vector in the direction of $2\vec{i} - \vec{j} - 2\vec{k}$

then
$$\hat{e} = \frac{2\vec{i} - \vec{j} - 2\vec{k}}{\sqrt{4+1+4}} = \frac{2\vec{i} - \vec{j} - 2\vec{k}}{\sqrt{9}} = \frac{2\vec{i} - \vec{j} - 2\vec{k}}{3}$$

$\therefore$ Directional derivative of ϕ along the given direction

$$D.D = \nabla\phi \cdot \hat{e}$$

$$= (8\vec{i} - \vec{j} - 10\vec{k}) \cdot \frac{(2\vec{i} - \vec{j} - 2\vec{k})}{3}$$

$$= \frac{1}{3} [16 + 1 + 20]$$

$$= \frac{37}{3}$$

8. Find the directional derivative of $\phi = 2xy + z^2$ at $(1, -1, 3)$ in the direction of $\vec{i} + 2\vec{j} + 3\vec{k}$.

9. Find the directional derivative of $\phi = xy^2 + yz^3$ at $(2, -1, 1)$ in the direction of $\vec{i} + 2\vec{j} + 2\vec{k}$.

Sol:- Given $\phi = 2xy + z^2$

$$\frac{\partial\phi}{\partial x} = 2y$$

$$\frac{\partial\phi}{\partial y} = 2x$$

$$\frac{\partial\phi}{\partial z} = 2z$$

$$\nabla\phi = 2y\vec{i} + 2x\vec{j} + 2z\vec{k}$$

at $(1, -1, 3)$

$$\nabla \phi = 2(-1)\vec{i} + 2(1)\vec{j} + 2(3)\vec{k} \\ = -2\vec{i} + 2\vec{j} + 6\vec{k}$$

Let $\hat{e}$ is the unit vector in the direction of $\vec{i} + 2\vec{j} + 3\vec{k}$
then $\hat{e} = \frac{-2\vec{i} + 2\vec{j} + 6\vec{k}}{\sqrt{1+4+9}}$

$$\hat{e} = \frac{-2\vec{i} + 2\vec{j} + 6\vec{k}}{\sqrt{14}} = \frac{-\vec{i} + \vec{j} + 3\vec{k}}{\sqrt{14}}$$

$$\therefore D \cdot D = \nabla \phi \cdot \hat{e}$$

$$= (-2\vec{i} + 2\vec{j} + 6\vec{k}) \cdot \frac{(-\vec{i} + \vec{j} + 3\vec{k})}{\sqrt{14}}$$

$$= \frac{-2+4+18}{\sqrt{14}} = \frac{20}{\sqrt{14}}$$

(9) soln- Let $\phi = xy^2 + yz^3$

$$\frac{\partial \phi}{\partial x} = y^2$$

$$\frac{\partial \phi}{\partial y} = x \cdot 2y + z^3$$

$$\frac{\partial \phi}{\partial z} = 3yz^2$$

$$\nabla \phi = y^2\vec{i} + (2xy + z^3)\vec{j} + 3yz^2\vec{k}$$

at $(2, -1, 1)$

$$\nabla \phi = (-1)^2\vec{i} + (2 \cdot 2(-1) + (1)^3)\vec{j} + 3 \cdot 2(-1)(1)^2\vec{k} \\ = \vec{i} - 3\vec{j} - 6\vec{k}$$

Let $\hat{e}$ is the unit vector in the direction of $\vec{i} + 2\vec{j} + 2\vec{k}$

$$\hat{e} = \frac{\vec{i} + 2\vec{j} + 2\vec{k}}{\sqrt{1+4+4}}$$

$$= \frac{\vec{i} + 2\vec{j} + 2\vec{k}}{3}$$

$$\therefore D \cdot D = \nabla \phi \cdot \hat{e}$$

$$= (\vec{i} - 3\vec{j} - 6\vec{k}) \cdot \frac{(\vec{i} + 2\vec{j} + 2\vec{k})}{3}$$

$$= \frac{1}{3} (1 - 6 - 6)$$

$$= \frac{-11}{3}$$

⑥ (i) $3x^2 - 2y^2 - 3z^2 + 8 = 0$ and $ax^2 + y^2 = bz$ at $(-1, 2, 1)$

Sol:- let

$$\phi = 3x^2 - 2y^2 - 3z^2 + 8 \quad \& \quad f = ax^2 + y^2 - bz$$

$$\nabla \phi = \bar{i} \frac{\partial \phi}{\partial x} + \bar{j} \frac{\partial \phi}{\partial y} + \bar{k} \frac{\partial \phi}{\partial z}$$

$$\frac{\partial \phi}{\partial x} = 6x$$

$$\frac{\partial \phi}{\partial y} = -4y$$

$$\frac{\partial \phi}{\partial z} = -6z$$

$$\nabla \phi = 6x\bar{i} - 4y\bar{j} - 6z\bar{k}$$

at point $(-1, 2, 1)$ $\nabla \phi$ is

$$\begin{aligned} \nabla \phi &= 6(-1)\bar{i} - 4(2)\bar{j} - 6(1)\bar{k} \\ &= -6\bar{i} - 8\bar{j} - 6\bar{k} = \bar{n}_1 \end{aligned}$$

let $f = ax^2 + y^2 - bz$

$$\frac{\partial f}{\partial x} = 2ax$$

$$\frac{\partial f}{\partial y} = 2y$$

$$\frac{\partial f}{\partial z} = -b$$

$$\nabla f = 2ax\bar{i} + 2y\bar{j} - b\bar{k}$$

at the point $(-1, 2, 1)$ ∇f is

$$\begin{aligned} \nabla f &= 2a(-1)\bar{i} + 2(2)\bar{j} - b\bar{k} \\ &= -2a\bar{i} + 4\bar{j} - b\bar{k} = \bar{n}_2 \end{aligned}$$

The vectors are orthogonal then $\bar{n}_1 \cdot \bar{n}_2 = 0$

$$(-6\bar{i} - 8\bar{j} - 6\bar{k}) \cdot (-2a\bar{i} + 4\bar{j} - b\bar{k}) = 0$$

$$12a - 32 + 6b = 0$$

$$12a + 6b = 32 \rightarrow (1)$$

Given that the point lies on the both the surfaces then

put $(-1, 2, 1)$ in 2nd surface

$$a(-1)^2 + (2)^2 = b(1)$$

$$a + 4 = b$$

$$a - b = -4 \rightarrow (2)$$

Solve (1) & (2)

$$12a + 6b = 32$$

$$(2) \times 6 \Rightarrow 6a - 6b = -24$$

$$18a = 8$$

$$a = \frac{8^4}{18^9}$$

$$a = \frac{4}{9}$$

from (2)

$$\frac{4}{9} - b = -4$$

$$\frac{4}{9} + 4 = b$$

$$\frac{40}{9} = b$$

$$\therefore a = \frac{4}{9}, \quad b = \frac{40}{9}$$

10. Find the directional derivative of $\phi = xy^2 + yz^3$ at the point $(2, -1, 1)$ in the direction of the normal to the surface

$$x \log z - y^2 + 4 = 0 \text{ at } (-1, 2, 1).$$

Sol:- Given $\phi = xy^2 + yz^3$

$$\frac{\partial \phi}{\partial x} = y^2$$

$$\frac{\partial \phi}{\partial y} = 2xy + z^3$$

$$\frac{\partial \phi}{\partial z} = 3yz^2$$

$$\text{grad } \phi = \nabla \phi = y^2 \bar{i} + (2xy + z^3) \bar{j} + 3yz^2 \bar{k}$$

at $(2, -1, 1)$

$$\begin{aligned} \nabla \phi &= (-1)^2 \bar{i} + (2(2)(-1) + (1)^3) \bar{j} + 3(-1)(1)^2 \bar{k} \\ &= \bar{i} - 3\bar{j} - 3\bar{k} \end{aligned}$$

$$\text{Let } f = x \log z - y^2 + 4$$

$$\frac{\partial f}{\partial x} = \log z$$

$$\frac{\partial f}{\partial y} = -2y$$

$$\frac{\partial f}{\partial z} = \frac{x}{z}$$

$$\nabla f = \log z \bar{i} - 2y \bar{j} + \frac{x}{z} \bar{k}$$

at $(-1, 2, 1)$

$$\begin{aligned} \nabla f &= \log(1) \bar{i} - 2(2) \bar{j} + \frac{-1}{1} \bar{k} \\ &= -4\bar{j} - \bar{k} \end{aligned}$$

' $\bar{n}$ ' is the normal to the surface f .

Let $\hat{e}$ be the unit normal vector.

$$\hat{e} = \frac{\bar{n}}{|\bar{n}|}$$

$$\hat{e} = \frac{0\bar{i} - 4\bar{j} - \bar{k}}{\sqrt{0+16+1}}$$

$$= \frac{0\bar{i} - 4\bar{j} - \bar{k}}{\sqrt{17}}$$

∴ Directional derivative along the given direction.

$$\begin{aligned} D.D &= \nabla \phi \cdot \hat{e} \\ &= 0\hat{i} - 3\hat{j} - 3\hat{k} \cdot \frac{0\hat{i} - 4\hat{j} - \hat{k}}{\sqrt{17}} \\ &= \frac{0+12+3}{\sqrt{17}} = \frac{15}{\sqrt{17}} \quad \parallel \end{aligned}$$

11. Find the directional derivative of the function

$\phi = xy^2 + yz^2 + zx^2$ along the tangent to the curve $x=t$, $y=t^2$, $z=t^3$ at the point $(1,1,1)$.

Sol:- Given $\phi = xy^2 + yz^2 + zx^2$

$$\frac{\partial \phi}{\partial x} = y^2 + 2zx \quad \frac{\partial \phi}{\partial y} = 2xy + z^2 \quad \frac{\partial \phi}{\partial z} = 2yz + x^2$$

$$\nabla \phi = (y^2 + 2zx)\hat{i} + (2xy + z^2)\hat{j} + (2yz + x^2)\hat{k}$$

at $(1,1,1)$

$$\begin{aligned} \nabla \phi &= (1^2 + 2(1)(1))\hat{i} + (2(1)(1) + 1^2)\hat{j} + (2(1)(1) + 1^2)\hat{k} \\ &= 3\hat{i} + 3\hat{j} + 3\hat{k} \end{aligned}$$

let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector of any point on the curve.

Given $x=t$, $y=t^2$, and $z=t^3$

$$\text{let } \vec{r} = t\hat{i} + t^2\hat{j} + t^3\hat{k}$$

$$\frac{d\vec{r}}{dt} = 1\hat{i} + 2t\hat{j} + 3t^2\hat{k}$$

$$\frac{d\vec{r}}{dt} \text{ at } (1,1,1) = \hat{i} + 2\hat{j} + 3\hat{k}$$

is the vector along the tangent to the curve.

let $\hat{e}$ be the unit tangent vector then.

$$\hat{e} = \frac{\frac{d\vec{r}}{dt}}{\left| \frac{d\vec{r}}{dt} \right|}$$

$$\begin{aligned} \hat{e} &= \frac{\hat{i} + 2\hat{j} + 3\hat{k}}{\sqrt{1+4+9}} \\ &= \frac{\hat{i} + 2\hat{j} + 3\hat{k}}{\sqrt{14}} \end{aligned}$$

∴ directional derivative along the given direction

$$D \cdot D = \nabla \phi \cdot \hat{e}$$

$$= 3\hat{i} + 3\hat{j} + 3\hat{k} \cdot \frac{\hat{i} + 2\hat{j} + 3\hat{k}}{\sqrt{14}}$$

$$= \frac{3+6+9}{\sqrt{14}}$$

$$= \frac{18}{\sqrt{14}} //$$

12. Find the directional derivative of the function $f = x^2 - y^2 + 2z^2$ at $P(1, 2, 3)$ in the direction of the line PQ where

$$Q = (5, 0, 4).$$

Sol:- Given $f = x^2 - y^2 + 2z^2$

$$\frac{\partial f}{\partial x} = 2x$$

$$\frac{\partial f}{\partial y} = -2y$$

$$\frac{\partial f}{\partial z} = 4z$$

$$\nabla f = 2x\hat{i} - 2y\hat{j} + 4z\hat{k}$$

at $(1, 2, 3)$

$$\nabla f = 2(1)\hat{i} - 2(2)\hat{j} + 4(3)\hat{k}$$

$$= 2\hat{i} - 4\hat{j} + 12\hat{k}$$

The position vectors of P and Q with respect to origin

$$\vec{OP} = \hat{i} + 2\hat{j} + 3\hat{k}$$

$$\vec{OQ} = 5\hat{i} + 0\hat{j} + 4\hat{k}$$

$$\therefore \vec{PQ} = \vec{OQ} - \vec{OP} \quad (\text{by triangular law})$$

$$= 5\hat{i} + 4\hat{k} - \hat{i} - 2\hat{j} - 3\hat{k}$$

let $\hat{e}$ be the

$$= 5\hat{i} + 4\hat{k} - \hat{i} - 2\hat{j} - 3\hat{k}$$

$$\vec{PQ} = 4\hat{i} - 2\hat{j} + \hat{k}$$

let $\hat{e}$ be the unit vector in the direction $\vec{PQ}$

$$\text{then } \hat{e} = \frac{\vec{PQ}}{|\vec{PQ}|}$$

$$= \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{16 + 4 + 1}}$$

$$= \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{21}}$$

Directional derivative along the given direction

$$D \cdot D = \nabla f \cdot \hat{e}$$

$$= (2\hat{i} - 4\hat{j} + 12\hat{k}) \cdot \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{21}}$$

$$D \cdot D = \frac{8+8+12}{\sqrt{21}}$$

$$= \frac{28}{\sqrt{21}} //$$

13. Find the directional derivative of $\nabla \cdot \nabla \phi$ at the point $(1, -2, 1)$ in the direction of the normal to the surface $xy^2z = 3x + z^2$ where $\phi = 2x^3y^2z^4$.

Sol:- Given $\phi = 2x^3y^2z^4$

$$\frac{\partial \phi}{\partial x} = 6x^2y^2z^4 \quad \frac{\partial \phi}{\partial y} = 4x^3yz^4 \quad \frac{\partial \phi}{\partial z} = 8x^3y^2z^3$$

$$\nabla \phi = 6x^2y^2z^4 \bar{i} + 4x^3yz^4 \bar{j} + 8x^3y^2z^3 \bar{k}$$

$$\text{let } \nabla \cdot \nabla \phi = \left[\bar{i} \frac{\partial}{\partial x} + \bar{j} \frac{\partial}{\partial y} + \bar{k} \frac{\partial}{\partial z} \right] \cdot \nabla \phi$$

$$= \left[\bar{i} \frac{\partial}{\partial x} + \bar{j} \frac{\partial}{\partial y} + \bar{k} \frac{\partial}{\partial z} \right] \cdot (6x^2y^2z^4 \bar{i} + 4x^3yz^4 \bar{j} + 8x^3y^2z^3 \bar{k})$$

$$= \frac{\partial}{\partial x} (6x^2y^2z^4) + \frac{\partial}{\partial y} (4x^3yz^4) + \frac{\partial}{\partial z} (8x^3y^2z^3)$$

$$= 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2 = f$$

$$\therefore f = 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$$

$$\frac{\partial f}{\partial x} = 12y^2z^4$$

$$\frac{\partial f}{\partial x} = 12y^2z^4 + 12x^2z^4 + 72x^2y^2z^2$$

$$\frac{\partial f}{\partial y} = 24xy^2z^4 + 48x^3yz^4$$

$$\frac{\partial f}{\partial z} = 48xy^2z^3 + 16x^3z^3 + 48x^3y^2z$$

$$\nabla \text{grad } f = (12y^2z^4 + 12x^2z^4 + 72x^2y^2z^2) \bar{i} + (24xy^2z^4 + 48x^3yz^4) \bar{j} + (48xy^2z^3 + 16x^3z^3 + 48x^3y^2z) \bar{k}$$

grad f at $(1, -2, 1)$

$$\text{grad } f = [12(-2)^2(1)^4 + 12(1)^2(1)^4 + 72(1)^2(-2)^2(1)^2] \bar{i} + [24(1)(-2)(1)^4 + 48(1)^3(-2)(1)^4] \bar{j} + [48(1)(-2)^3(1)^3 + 16(1)(1)^3 + 48(1)^3(-2)^3(1)] \bar{k}$$

$$\nabla f = (12+12+288)\bar{i} + (-48-96)\bar{j} + (192+16+192)\bar{k}$$

$$\nabla f = 348\bar{i} - 144\bar{j} + 400\bar{k}$$

$$\text{let } g = xy^2z - 3x - z^2$$

$$\frac{\partial g}{\partial x} = y^2z - 3$$

$$\frac{\partial g}{\partial y} = 2xyz$$

$$\frac{\partial g}{\partial z} = xy^2 - 2z$$

$$\nabla g = (y^2z - 3)\bar{i} + 2xyz\bar{j} + (xy^2 - 2z)\bar{k}$$

$$\nabla g \text{ at } (1, -2, 1)$$

$$\nabla g = ((-2)^2(1) - 3)\bar{i} + 2(1)(-2)(1)\bar{j} + ((1)(-2)^2 - 2(1))\bar{k}$$

$$= (4 - 3)\bar{i} - 4\bar{j} + (4 - 2)\bar{k}$$

$$= \bar{i} - 4\bar{j} + 2\bar{k}$$

let $\hat{e}$ be the Unit ^{normal} vector then

$$\hat{e} = \frac{\nabla g}{|\nabla g|}$$

$$= \frac{\bar{i} - 4\bar{j} + 2\bar{k}}{\sqrt{1+16+4}}$$

$$= \frac{\bar{i} - 4\bar{j} + 2\bar{k}}{\sqrt{21}}$$

$\therefore$ Directional derivative along the given direction

$$D \cdot D = \nabla f \cdot \hat{e}$$

$$= 348\bar{i} - 144\bar{j} + 400\bar{k} \cdot \left(\frac{\bar{i} - 4\bar{j} + 2\bar{k}}{\sqrt{21}} \right)$$

$$= \frac{348 + 576 + 800}{\sqrt{21}}$$

$$= \frac{1724}{\sqrt{21}}$$

14. Show that $\nabla(f(r)) = \frac{f'(r)}{r} \bar{r}$, where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

Sol:- Given

$$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$$

$$|\bar{r}| = r = \sqrt{x^2 + y^2 + z^2}$$

S.O.B.S

$$r^2 = x^2 + y^2 + z^2 \rightarrow \textcircled{1}$$

diff. eqn ① partially w.r.t 'x' $\therefore (2xy + 2yz + 2xz) = 1$

$$2y \frac{\partial r}{\partial x} = 2x$$

$$y \frac{\partial r}{\partial x} = x$$

$$\frac{\partial r}{\partial x} = \frac{x}{y}$$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{x}, \frac{\partial r}{\partial z} = \frac{z}{x}$

Now $\nabla[f(r)] = \left[\bar{i} \frac{\partial}{\partial x} + \bar{j} \frac{\partial}{\partial y} + \bar{k} \frac{\partial}{\partial z} \right] f(r)$

$$= \bar{i} \frac{\partial}{\partial x} f(r) + \bar{j} \frac{\partial}{\partial y} f(r) + \bar{k} \frac{\partial}{\partial z} f(r)$$

$$= \bar{i} f'(r) \frac{\partial r}{\partial x} + \bar{j} f'(r) \frac{\partial r}{\partial y} + \bar{k} f'(r) \frac{\partial r}{\partial z}$$

$$= \bar{i} f'(r) \frac{x}{y} + \bar{j} f'(r) \frac{y}{x} + \bar{k} f'(r) \frac{z}{x}$$

$$= \frac{f'(r)}{r} [x\bar{i} + y\bar{j} + z\bar{k}]$$

$$\nabla[f(r)] = \frac{f'(r)}{r} \cdot \bar{r}$$

$$\Rightarrow \Delta[f(r)] = \frac{f'(r)}{r} \cdot \bar{r}$$

(i) $\nabla[\log r] = \frac{\bar{r}}{r^2}$

(ii) $\nabla[r^n] = n r^{n-2} \bar{r}$

(iii) $\nabla\left[\frac{1}{r}\right] = \frac{-\bar{r}}{r^3}$

(iv) $\nabla(r) = \frac{\bar{r}}{r}$

15. Find the greatest value of the directional derivative of the function $f = x^2 y z^3$ at $(2, 1, -1)$.

Sol:- Given $f = x^2 y z^3$

$$\frac{\partial f}{\partial x} = 2x y z^3$$

We know that the greatest value of directional derivative of 'f' at a point is equal to magnitude of $\text{grad } f$ at that point.

$$\frac{\partial f}{\partial x} = 2xyz^3 \quad \frac{\partial f}{\partial y} = x^2z^3 \quad \frac{\partial f}{\partial z} = 3x^2yz^2$$

$$\nabla f = 2xyz^3 \bar{i} + x^2z^3 \bar{j} + 3x^2yz^2 \bar{k}$$

at (2, 1, -1)

$$\nabla f = 2(2)(1)(-1)^3 \bar{i} + (2)^2(-1)^3 \bar{j} + 3(2)^2(1)(-1)^2 \bar{k}$$

$$= -4\bar{i} - 2\bar{j} + 12\bar{k}$$

$$\therefore |\text{Grad } f| = \sqrt{16 + 4 + 144}$$

$$= \sqrt{164}$$

(i) $\nabla(\log r) = \frac{\bar{r}}{r^2}$

Sol:- Given $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$|\bar{r}| = r = \sqrt{x^2 + y^2 + z^2}$$

S.O.B.S

$$r^2 = x^2 + y^2 + z^2 \rightarrow (1)$$

diff eqn (1) partially w.r.t 'x'

$$2r \frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$

$$\text{Now } \nabla(\log r) = \left[\bar{i} \frac{\partial}{\partial x} + \bar{j} \frac{\partial}{\partial y} + \bar{k} \frac{\partial}{\partial z} \right] \log r$$

$$= \bar{i} \frac{\partial}{\partial x}(\log r) + \bar{j} \frac{\partial}{\partial y}(\log r) + \bar{k} \frac{\partial}{\partial z}(\log r)$$

$$= \bar{i} \frac{1}{r} \frac{\partial r}{\partial x} + \bar{j} \frac{1}{r} \frac{\partial r}{\partial y} + \bar{k} \frac{1}{r} \frac{\partial r}{\partial z}$$

$$= \bar{i} \frac{1}{r} \cdot \frac{x}{r} + \bar{j} \frac{1}{r} \cdot \frac{y}{r} + \bar{k} \frac{1}{r} \cdot \frac{z}{r}$$

$$= \frac{1}{r^2} [x\bar{i} + y\bar{j} + z\bar{k}]$$

$$\nabla(\log r) = \frac{\bar{r}}{r^2}$$

(ii) $\nabla(r^n) = n r^{n-2} \bar{r}$

Sol:- Given $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$|\bar{r}| = r = \sqrt{x^2 + y^2 + z^2}$$

S.O.B.S

$$r^2 = x^2 + y^2 + z^2 \rightarrow (1)$$

diff eqn (1) partially w.r.t 'x'

$$2r \frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

Now $\nabla(r^n) = \left(\bar{i} \frac{\partial}{\partial x} + \bar{j} \frac{\partial}{\partial y} + \bar{k} \frac{\partial}{\partial z} \right) r^n$
 $= \bar{i} \frac{\partial}{\partial x} r^n + \bar{j} \frac{\partial}{\partial y} r^n + \bar{k} \frac{\partial}{\partial z} r^n$
 $= \bar{i} n \cdot r^{n-1} \frac{\partial r}{\partial x} + \bar{j} n \cdot r^{n-1} \frac{\partial r}{\partial y} + \bar{k} n \cdot r^{n-1} \frac{\partial r}{\partial z}$
 $= \bar{i} n \cdot r^{n-1} \frac{x}{r} + \bar{j} n \cdot r^{n-1} \frac{y}{r} + \bar{k} n \cdot r^{n-1} \frac{z}{r}$
 $= \frac{n \cdot r^{n-1}}{r} (x\bar{i} + y\bar{j} + z\bar{k})$

$$\nabla(r^n) = \frac{n \cdot r^{n-1}}{r} \cdot \bar{r}$$

$$\nabla(r^n) = n \cdot r^{n-1} \cdot \bar{r}$$

$$\nabla(r^n) = n \cdot r^{n-2} \cdot \bar{r} //$$

(iii) $\nabla\left(\frac{1}{r}\right) = \frac{-\bar{r}}{r^3}$

Sol:- $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$
 $|\bar{r}| = r = \sqrt{x^2 + y^2 + z^2}$
 $r^2 = x^2 + y^2 + z^2$

diff. w.r. to 'x' partially

$$2r \frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

Now $\nabla\left(\frac{1}{r}\right) = \left(\bar{i} \frac{\partial}{\partial x} + \bar{j} \frac{\partial}{\partial y} + \bar{k} \frac{\partial}{\partial z} \right) \frac{1}{r}$

$$= \bar{i} \frac{\partial}{\partial x} \left(\frac{1}{r} \right) + \bar{j} \frac{\partial}{\partial y} \left(\frac{1}{r} \right) + \bar{k} \frac{\partial}{\partial z} \left(\frac{1}{r} \right)$$

$$= \bar{i} \left(\frac{-1}{r^2} \right) \frac{\partial r}{\partial x} + \bar{j} \left(\frac{-1}{r^2} \right) \frac{\partial r}{\partial y} + \bar{k} \left(\frac{-1}{r^2} \right) \frac{\partial r}{\partial z}$$

$$= \frac{-1}{r^2} \bar{i} \frac{x}{r} + \bar{j} \left(\frac{-1}{r^2} \right) \frac{y}{r} + \bar{k} \left(\frac{-1}{r^2} \right) \frac{z}{r}$$

$$= \frac{-1}{r^3} (x\bar{i} + y\bar{j} + z\bar{k})$$

$$\nabla\left(\frac{1}{r}\right) = \frac{-\bar{r}}{r^3} //$$

(iv) $\nabla(r) = \frac{\bar{r}}{r}$

Sol:- $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$|\bar{r}| = r = \sqrt{x^2 + y^2 + z^2}$$

$$r^2 = x^2 + y^2 + z^2$$

diff w.r.t 'x' partially

$$2r \frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

$$\text{Now } \nabla(r) = \left[\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right] (r)$$

$$= \hat{i} \frac{\partial}{\partial x} r + \hat{j} \frac{\partial}{\partial y} r + \hat{k} \frac{\partial}{\partial z} r$$

$$= \hat{i} \frac{x}{r} + \hat{j} \frac{y}{r} + \hat{k} \frac{z}{r}$$

$$= \frac{1}{r} (x\hat{i} + y\hat{j} + z\hat{k})$$

$$\nabla(r) = \frac{1}{r} \cdot \vec{r}$$

16. The temperature at a point (x, y, z) is given by $T(x, y, z) = x^2 + y^2 - z$.
a mosquito is located at $(1, 1, 2)$ desires to fly in such a direction that will get warm as soon as possible in what direction should it fly.

Sol:- Given $T = x^2 + y^2 - z$

$$\nabla T = 2x\hat{i} + 2y\hat{j} - \hat{k}$$

$$\nabla T \text{ at } (1, 1, 2)$$

$$\nabla T = 2\hat{i} + 2\hat{j} - \hat{k}$$

The mosquito should fly in the direction of normal to the surface, i.e., ∇T at $(1, 1, 2)$.

Hence the mosquito should fly in the direction of

$$2\hat{i} + 2\hat{j} - \hat{k}$$

Divergence of a vector point function:-

if $\vec{f} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ is a continuously differentiable

vector point function. Then $\nabla \cdot \vec{f} = \left[\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right] (f_1\hat{i} + f_2\hat{j} + f_3\hat{k})$

$$\nabla \cdot \vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \text{ is called the}$$

divergence of $\vec{f}$ and is denoted by 'div $\vec{f}$ '.

$$\text{Hence } \text{div } \vec{f} = \nabla \cdot \vec{f}$$

Note :-

* $\nabla \cdot \vec{f}$ is a scalar point function.

* $\text{div } \vec{f}$ can also be written as

$$\text{div } \vec{f} = \vec{i} \cdot \frac{\partial \vec{f}}{\partial x} + \vec{j} \cdot \frac{\partial \vec{f}}{\partial y} + \vec{k} \cdot \frac{\partial \vec{f}}{\partial z}$$

$$= \sum \vec{i} \cdot \frac{\partial \vec{f}}{\partial x}$$

Solenoidal vector (or) Incompressible vector :-

A vector point function $\vec{f}$ is said to be solenoidal

$$\text{if } \text{div } \vec{f} = 0$$

1. Evaluate divergence of $2x^2z\vec{i} + xy^2z\vec{j} + 3yz^2\vec{k}$ at the point $(1, 1, 1)$.

Sol:- Let $\vec{f} = 2x^2z\vec{i} + xy^2z\vec{j} + 3yz^2\vec{k}$

$$\vec{f} = f_1\vec{i} + f_2\vec{j} + f_3\vec{k}$$

$$\text{Then } \text{div } \vec{f} = \nabla \cdot \vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= \frac{\partial}{\partial x} (2x^2z) + \frac{\partial}{\partial y} (xy^2z) + \frac{\partial}{\partial z} (3yz^2)$$

$$= \frac{\partial}{\partial x} 4xz - 2y^2zx + 6yz$$

$\text{div } \vec{f}$ at $(1, 1, 1)$

$$\text{div } \vec{f} = 4(1)(1) - 2(1)(1)(1) + 6(1)(1)$$

$$= 4 - 2 + 6$$

$$= 8 //$$

(i) If $\vec{f} = xy^2\vec{i} + 2x^2yz\vec{j} - 3yz^2\vec{k}$ find $\text{div } \vec{f}$ at $(1, -1, 1)$

Sol:- Let $\vec{f} = xy^2\vec{i} + 2x^2yz\vec{j} - 3yz^2\vec{k}$

$$\text{div } \vec{f} = \nabla \cdot \vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= \frac{\partial}{\partial x} (xy^2) + \frac{\partial}{\partial y} (2x^2yz) + \frac{\partial}{\partial z} (-3yz^2)$$

$$= y^2 + 2x^2z - 6yz$$

$\text{div } \vec{f}$ at $(1, -1, 1)$

$$\text{div } \vec{f} = (-1)^2 + 2(1)^2(1) - 6(-1)(1)$$

$$= 1 + 2 + 6$$

$$= 9 //$$

(i) Find $\nabla \cdot (e^y \sin x \cos z \bar{i} + e^x \sin y \cos z \bar{j} + z^2 e^z \bar{k})$.

Sol:- let $\vec{f} = e^y \sin x \cos z \bar{i} + e^x \sin y \cos z \bar{j} + z^2 e^z \bar{k}$

$$\text{div } \vec{f} = \nabla \cdot \vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$\nabla \cdot \vec{f} = \frac{\partial}{\partial x} (e^y \sin x \cos z) + \frac{\partial}{\partial y} (e^x \sin y \cos z) +$$

$$\frac{\partial}{\partial z} (z^2 e^z)$$

$$\nabla \cdot \vec{f} = e^y \cos x \cos z + e^x \cos y \cos z + 2ze^z + z^2 e^z$$

$$\nabla \cdot \vec{f} = e^y \cos x \cos z + e^x \cos y \cos z + 2ze^z + z^2 e^z //$$

2. Find $\text{div } \vec{f}$, where $\vec{f} = \text{grad} (x^3 + y^3 + z^3 - 3xyz)$

Sol:- let $\vec{f} = \text{grad } \phi$, where $\phi = x^3 + y^3 + z^3 - 3xyz$

$$= \bar{i} \frac{\partial \phi}{\partial x} + \bar{j} \frac{\partial \phi}{\partial y} + \bar{k} \frac{\partial \phi}{\partial z}$$

$$\vec{f} = (3x^2 - 3yz) \bar{i} + (3y^2 - 3xz) \bar{j} + (3z^2 - 3xy) \bar{k}$$

$$\vec{f} = f_1 \bar{i} + f_2 \bar{j} + f_3 \bar{k}$$

$$\text{div } \vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= \frac{\partial}{\partial x} (3x^2 - 3yz) + \frac{\partial}{\partial y} (3y^2 - 3xz) + \frac{\partial}{\partial z} (3z^2 - 3xy)$$

$$= 6x + 6y + 6z$$

$$\text{div } \vec{f} = 6(x + y + z) //$$

(i) If $\phi = 2x^3 y^2 z^4$ then find $\text{div}(\text{grad } \phi)$

Sol:- let $\phi = 2x^3 y^2 z^4$

$$\text{grad } \phi = 6x^2 y^2 z^4 \bar{i} + 4x^3 y z^4 \bar{j} + 8x^3 y^2 z^3 \bar{k}$$

$$\text{then } \text{div}(\text{grad } \phi) = \frac{\partial}{\partial x} (6x^2 y^2 z^4) + \frac{\partial}{\partial y} (4x^3 y z^4) + \frac{\partial}{\partial z} (8x^3 y^2 z^3)$$

$$= 12xy^2z^4 + 12x^3yz^4$$

$$= 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2 //$$

3. Show that $\vec{A} = 3y^4 z^2 \bar{i} + ux^3 z^2 \bar{j} - 3x^2 y^2 \bar{k}$ is solenoidal vector.

Sol:- Given $\vec{A} = 3y^4 z^2 \bar{i} + ux^3 z^2 \bar{j} - 3x^2 y^2 \bar{k}$

$$\vec{A} = A_1 \bar{i} + A_2 \bar{j} + A_3 \bar{k}$$

(iv) $\vec{A} = 3xy^2z^2\vec{i} + 2xy^3\vec{j} - x^2yz\vec{k}$ and $f = 3x^2 - yz$

$$\text{grad } f = \vec{i} \frac{\partial f}{\partial x} + \vec{j} \frac{\partial f}{\partial y} + \vec{k} \frac{\partial f}{\partial z}$$

$$\frac{\partial f}{\partial x} = 6x \quad \frac{\partial f}{\partial y} = -z \quad \frac{\partial f}{\partial z} = -y$$

$$\text{grad } f = \nabla f = 6x\vec{i} - z\vec{j} - y\vec{k}$$

$$\text{Now } \nabla \cdot \nabla f = \frac{\partial}{\partial x} 6x - \frac{\partial}{\partial y} (z) - \frac{\partial}{\partial z} y$$

$$= 6 - 0 + 0$$

$$\nabla \cdot \nabla f = 6$$

Curl of a vector point function :-

If ' $\vec{f}$ ' is any differentiable ^{vector point} function. Then the vector function is defined by

$$\vec{i} \times \frac{\partial \vec{f}}{\partial x} + \vec{j} \times \frac{\partial \vec{f}}{\partial y} + \vec{k} \times \frac{\partial \vec{f}}{\partial z}$$

$$= \sum \vec{i} \times \frac{\partial \vec{f}}{\partial x} \text{ is called curl of a}$$

vector point function and is denoted by $\text{curl } \vec{f}$ (or) $\nabla \times \vec{f}$

If $\vec{f} = f_1\vec{i} + f_2\vec{j} + f_3\vec{k}$ then

$$\text{curl } \vec{f} = \nabla \times \vec{f} = \left[\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right] \times (f_1\vec{i} + f_2\vec{j} + f_3\vec{k})$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

Irrrotational vector :-

A vector point function $\vec{f}$ is said to be irrotational if $\text{curl } \vec{f} = \vec{0}$.

Note :-

$$* \text{ If } \text{curl}(\vec{A} \pm \vec{B}) = \text{curl } \vec{A} \pm \text{curl } \vec{B}$$

1. determine the $\text{curl}(xyz^2\vec{i} + yzx^2\vec{j} + zxy^2\vec{k})$ at the point $(1, 2, 3)$

Sol:- let $\vec{f} = xyz^2\vec{i} + yzx^2\vec{j} + zxy^2\vec{k}$

$$\text{curl } \vec{f} = \nabla \times \vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz^2 & yzx^2 & zxy^2 \end{vmatrix}$$

$$= \vec{i} \left(\frac{\partial}{\partial y}(zxy^2) - \frac{\partial}{\partial z}(yzx^2) \right) - \vec{j} \left(\frac{\partial}{\partial x}(zxy^2) - \frac{\partial}{\partial z}(xyz^2) \right) + \vec{k} \left(\frac{\partial}{\partial x}(yzx^2) - \frac{\partial}{\partial y}(xyz^2) \right)$$

$$= \vec{i}(2xyz - x^2y) - \vec{j}(zy^2 - 2xyz) + \vec{k}(2xyz - x^2y)$$

$\text{curl } \vec{f}$ at $(1, 2, 3)$

$$\text{curl } \vec{f} = \vec{i}(2(1)(2)(3) - (1)^2(2)) - \vec{j}(3(2)^2 - 2(1)(2)(3)) + \vec{k}(2(1)(2)(3) - (1)^2(2))$$

$$= \vec{i}(12 - 2) - \vec{j}(12 - 12) + \vec{k}(12 - 2)$$

$$= 10\vec{i} + 3\vec{k}$$

(i) If $\vec{f} = xy^2\vec{i} + 2x^2yz\vec{j} - 3yz^2\vec{k}$ find $\text{curl } \vec{f}$ at $(1, -1, 1)$

Sol:- $\vec{f} = xy^2\vec{i} + 2x^2yz\vec{j} - 3yz^2\vec{k}$

$$\text{curl } \vec{f} = \nabla \times \vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix}$$

$$= \vec{i} \left(\frac{\partial}{\partial y}(-3yz^2) - \frac{\partial}{\partial z}(2x^2yz) \right) - \vec{j} \left(\frac{\partial}{\partial x}(-3yz^2) - \frac{\partial}{\partial z}(xy^2) \right)$$

$$+ \vec{k} \left(\frac{\partial}{\partial x}(2x^2yz) - \frac{\partial}{\partial y}(xy^2) \right)$$

$$= \vec{i}(-3z^2 - 2x^2y) - \vec{j}(0 - 0) + \vec{k}(4xy - 2xy)$$

at $(1, -1, 1)$

$$\text{curl } \vec{f} = \vec{i}(-3(1)^2 - 2(1)^2(-1)) + \vec{k}(4(1)(-1) - 2(1)(-1))$$

$$\text{curl } \vec{f} = \vec{i}(-3+2) + \vec{k}(-4+2)$$

$$= -\vec{i} - 2\vec{k}$$

2. Show that the vector $\vec{f} = e^{x+y+z}(\vec{i} + \vec{j} + \vec{k})$ is irrotational

Sol:- Given $\vec{f} = e^{x+y+z}(\vec{i} + \vec{j} + \vec{k})$

$$= e^{x+y+z} \vec{i} + e^{x+y+z} \vec{j} + e^{x+y+z} \vec{k}$$

$$\text{curl } \vec{f} = \nabla \times \vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{x+y+z} & e^{x+y+z} & e^{x+y+z} \end{vmatrix}$$

$$= \vec{i} \left[\frac{\partial}{\partial y} (e^{x+y+z}) - \frac{\partial}{\partial z} (e^{x+y+z}) \right] - \vec{j} \left[\frac{\partial}{\partial x} (e^{x+y+z}) - \frac{\partial}{\partial z} (e^{x+y+z}) \right]$$

$$+ \vec{k} \left[\frac{\partial}{\partial x} (e^{x+y+z}) - \frac{\partial}{\partial y} (e^{x+y+z}) \right]$$

$$= \vec{i} [e^{x+y+z} - e^{x+y+z}] - \vec{j} [e^{x+y+z} - e^{x+y+z}] + \vec{k} [e^{x+y+z} - e^{x+y+z}]$$

$$= \vec{0}$$

$$\text{curl } \vec{f} = \vec{0}$$

$\Rightarrow \vec{f}$ is a irrotational

3. Let $\vec{f} = (x+y+1)\vec{i} + \vec{j} - (x+y)\vec{k}$ then show that $\vec{f} \cdot \text{curl } \vec{f} = 0$.

Sol:- Given $\vec{f} = (x+y+1)\vec{i} + \vec{j} - (x+y)\vec{k}$

$$\text{curl } \vec{f} = \nabla \times \vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y+1 & 1 & -x-y \end{vmatrix}$$

$$= \vec{i} \left[\frac{\partial}{\partial y} (-x-y) - \frac{\partial}{\partial z} (1) \right] - \vec{j} \left[\frac{\partial}{\partial x} (-x-y) - \frac{\partial}{\partial z} (x+y+1) \right]$$

$$+ \vec{k} \left[\frac{\partial}{\partial x} (1) - \frac{\partial}{\partial y} (x+y+1) \right]$$

$$= \vec{i} (-1-0) - \vec{j} (-1-0) + \vec{k} (0 - (1+0))$$

$$\text{curl } \vec{f} = -\vec{i} + \vec{j} - \vec{k}$$

$$\vec{f} \cdot \text{curl } \vec{f} = [(x+y+1)\vec{i} + \vec{j} - (x+y)\vec{k}] \cdot [-\vec{i} + \vec{j} - \vec{k}]$$

$$= (x+y+1)(-1) + 1+x+y$$

$$= -x-y-1+1+x+y$$

$$\vec{F} \cdot \text{curl} \vec{F} = 0$$

4. Find the constant a, b, c such that $\vec{F} = (2x+3y+az)\vec{i} + (bx+2y+3z)\vec{j} + (2x+cy+3z)\vec{k}$ is irrotational.

Sol:- Given $\vec{F} = (2x+3y+az)\vec{i} + (bx+2y+3z)\vec{j} + (2x+cy+3z)\vec{k}$

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+3y+az & bx+2y+3z & 2x+cy+3z \end{vmatrix}$$

$$= \vec{i} \left[\frac{\partial}{\partial y} (2x+cy+3z) - \frac{\partial}{\partial z} (bx+2y+3z) \right] - \vec{j} \left[\frac{\partial}{\partial x} (2x+cy+3z) - \frac{\partial}{\partial z} (2x+3y+az) \right] + \vec{k} \left[\frac{\partial}{\partial x} (bx+2y+3z) - \frac{\partial}{\partial y} (2x+3y+az) \right]$$

$$= \vec{i} (c-3) - \vec{j} (2-a) + \vec{k} (b-3)$$

$$= (c-3)\vec{i} - (2-a)\vec{j} + (b-3)\vec{k}$$

$\therefore \vec{F}$ is irrotational then $\text{curl } \vec{F} = 0$

$$(c-3)\vec{i} - (2-a)\vec{j} + (b-3)\vec{k} = 0\vec{i} + 0\vec{j} + 0\vec{k}$$

equating the components of $\vec{i}, \vec{j}, \vec{k}$

$$c-3=0, \quad -2+a=0, \quad b-3=0$$

$$c=3, \quad a=2, \quad b=3$$

$$\therefore a=2, b=3, c=3$$

Note :-

* If $\text{curl } \vec{F} = 0$ then there exist a scalar point function ϕ .

Such that $\vec{F} = \nabla \phi$. Where ϕ is called scalar potential of $\vec{F}$ and $\vec{F}$ is conservative.

1. Prove that $\vec{A} = (6xy+z^3)\vec{i} + (3x^2-z)\vec{j} + (3xz^2-y)\vec{k}$ is irrotational find a scalar potential function ϕ . Such that $\vec{A} = \nabla \phi$.

Sol:- Given $\vec{A} = (6xy+z^3)\vec{i} + (3x^2-z)\vec{j} + (3xz^2-y)\vec{k}$

$$\text{curl } \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (6xy+z^3) & (3x^2z) & (3xz^2y) \end{vmatrix}$$

$$= \vec{i} \left(\frac{\partial}{\partial y} (3xz^2y) - \frac{\partial}{\partial z} (3x^2z) \right) - \vec{j} \left(\frac{\partial}{\partial x} (3xz^2y) - \frac{\partial}{\partial z} (6xy+z^3) \right)$$

$$+ \vec{k} \left(\frac{\partial}{\partial x} (3x^2z) - \frac{\partial}{\partial y} (6xy+z^3) \right)$$

$$= \vec{i} (-1+1) - \vec{j} (3z^2-3z^2) + \vec{k} (6x-6x)$$

$$= 0\vec{i} + 0\vec{j} + 0\vec{k}$$

$$\text{curl } \vec{A} = 0$$

$\therefore \vec{A}$ is irrotational

$$\text{Given } \vec{A} = \nabla f$$

$$(6xy+z^3)\vec{i} + (3x^2z)\vec{j} + (3xz^2y)\vec{k} = \vec{i} \frac{\partial f}{\partial x} + \vec{j} \frac{\partial f}{\partial y} + \vec{k} \frac{\partial f}{\partial z}$$

Method 1 :-

equating the components of $\vec{i}, \vec{j}, \vec{k}$

$$\frac{\partial f}{\partial x} = 6xy + z^3 \rightarrow (1)$$

$$\frac{\partial f}{\partial y} = 3x^2z \rightarrow (2)$$

$$\frac{\partial f}{\partial z} = 3xz^2y \rightarrow (3)$$

Integrating eqn (1) w.r.t 'x'

$$f = \int (6xy + z^3) dx$$

$$= 6y \frac{x^2}{2} + z^3x + C_1(y, z)$$

$$f = 3x^2y + xz^3 + C_1(y, z) \rightarrow (4)$$

diff eqn (4) partially w.r.t 'y'

$$\frac{\partial f}{\partial y} = 3x^2 + 0 + \frac{\partial C_1}{\partial y}$$

$$3x^2z = 3x^2 + \frac{\partial C_1}{\partial y}$$

$$\frac{\partial C_1}{\partial y} = -z \rightarrow (5)$$

integrating eqn (5) partially w.r.t to 'y'

$$C_1 = -zy/dy$$

$$C_1 = -zy + C_2(z) \rightarrow (6)$$

Substitute eqn (6) in (4)

$$f = 3x^2y + xz^3 - yz + c_2(z) \rightarrow (7)$$

diff. eqn (7) partially w.r.t 'z'

$$\frac{\partial f}{\partial z} = 0 + 3xz^2 - y + \frac{\partial c_2}{\partial z}$$

$$3xz^2 - y = 3xz^2 - y + \frac{\partial c_2}{\partial z}$$

$$\frac{\partial c_2}{\partial z} = 0$$

$$c_2(z) = k \quad (\because k = \text{constant})$$

$$\therefore \text{The required } f = 3x^2y + xz^3 - yz + k$$

Method 2:-

integrating eqn's (1), (2), (3) partially w.r.t x, y, z. we get

$$f = 3x^2y + xz^3 + c_1(y, z)$$

$$f = 3x^2y - yz + c_2(x, y)$$

$$f = 3xz^3 - yz + c_3(x, y)$$

$$\therefore \text{Repeated } f = 3x^2y - yz + xz^3 + k //$$

2. Show that the vector field $\vec{F} = 2xyyz^2\vec{i} + (x^2z^2 + z\cos(yz))\vec{j} + (2x^2yz + y\cos(yz))\vec{k}$ is irrotational and find its scalar potential.

Sol:- Given $\vec{F} = 2xyyz^2\vec{i} + (x^2z^2 + z\cos(yz))\vec{j} + (2x^2yz + y\cos(yz))\vec{k}$

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyyz^2 & x^2z^2 + z\cos(yz) & 2x^2yz + y\cos(yz) \end{vmatrix}$$

$$= \vec{i} \left[\frac{\partial}{\partial y} (2x^2yz + y\cos(yz)) - \frac{\partial}{\partial z} (x^2z^2 + z\cos(yz)) \right] - \vec{j} \left[\frac{\partial}{\partial x} (2x^2yz + y\cos(yz)) - \frac{\partial}{\partial z} (2xyyz^2) \right] + \vec{k} \left[\frac{\partial}{\partial x} (x^2z^2 + z\cos(yz)) - \frac{\partial}{\partial y} (2xyyz^2) \right]$$

$$= \vec{i} [2x^2z + y\cos(yz) - \cos(yz)] - \vec{j} [2x^2z + z\cos(yz) - 2xyyz^2] + \vec{k} [2xz^2 - 2xz^2]$$

$$\text{curl } \vec{F} = 0$$

$\therefore \text{curl } \vec{F} = 0$ then $\exists$ scalar point function ϕ such that

$$\vec{F} = \nabla \phi$$

$$2xyz^2 \vec{i} + (x^2yz^2 + z \cos(yz)) \vec{j} + (2x^2yz + y \cos(yz)) \vec{k} = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}$$

in equating the components of $\vec{i}, \vec{j}, \vec{k}$

$$\frac{\partial \phi}{\partial x} = 2xyz^2 \rightarrow (1) \quad \frac{\partial \phi}{\partial y} = x^2z^2 + z \cos(yz) \rightarrow (2) \quad \frac{\partial \phi}{\partial z} = 2x^2yz + y \cos(yz) \rightarrow (3)$$

integrating eqns (1), (2), (3) partially w.r.t x, y, z we get

$$(1) \Rightarrow \phi = \frac{x^2}{2} y z^2 + c_1(y, z)$$

$$(2) \Rightarrow \phi = x^2 z^2 \int dy + z \int \cos(yz) dy$$

$$= x^2 z^2 y + z \frac{\sin(yz)}{z}$$

$$\phi = x^2 z^2 y + \sin(yz) + c_2(x, z)$$

$$(3) \Rightarrow \phi = x^2 y \int \frac{z^2}{z} dz + y \int \cos(yz) dz$$

$$= x^2 y z^2 + \sin(yz) \frac{1}{y}$$

$$\phi = x^2 y z^2 + \sin(yz) + c_3(x, y)$$

$$\therefore \text{Repeated } \phi = x^2 y z^2 + \sin(yz) + k$$

3. Prove that $\text{curl}(\vec{r}) = \text{curl}(r^n \vec{r}) = 0$ where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

If $\vec{r}$ is the position vector of any point in space, then $(r^n \vec{r})$ is irrotational.

Sol:- Given $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$|\vec{r}| = r = \sqrt{x^2 + y^2 + z^2}$$

S.O.B.S

$$r^2 = x^2 + y^2 + z^2 \rightarrow (1)$$

diff (1) partially w.r.t 'x'

$$2r \frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

Let $\vec{r} = r^n \vec{r}$

$$= r^n (x\vec{i} + y\vec{j} + z\vec{k})$$

$$= r^n x\vec{i} + r^n y\vec{j} + r^n z\vec{k}$$

$$f_1 \vec{i} + f_2 \vec{j} + f_3 \vec{k}$$

$$\text{curl } \vec{r} = \nabla \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^n x & r^n y & r^n z \end{vmatrix}$$

$$= \vec{i} \left(\frac{\partial}{\partial y} r^n z - \frac{\partial}{\partial z} r^n y \right) - \vec{j} \left(\frac{\partial}{\partial x} r^n z - \frac{\partial}{\partial z} r^n x \right) + \vec{k} \left(\frac{\partial}{\partial x} r^n y - \frac{\partial}{\partial y} r^n x \right)$$

$$= \vec{i} \left(z n r^{n-1} \frac{\partial r}{\partial y} - y n r^{n-1} \frac{\partial r}{\partial z} \right) - \vec{j} \left(z n r^{n-1} \frac{\partial r}{\partial x} - x n r^{n-1} \frac{\partial r}{\partial z} \right) + \vec{k} \left(y n r^{n-1} \frac{\partial r}{\partial x} - x n r^{n-1} \frac{\partial r}{\partial y} \right)$$

$$= \vec{i} \left(z n r^{n-1} \frac{y}{r} - y n r^{n-1} \frac{z}{r} \right) - \vec{j} \left(z n r^{n-1} \frac{x}{r} - x n r^{n-1} \frac{z}{r} \right) + \vec{k} \left(y n r^{n-1} \frac{x}{r} - x n r^{n-1} \frac{y}{r} \right)$$

$$= 0\vec{i} + 0\vec{j} + 0\vec{k}$$

$\text{curl } \vec{r} = 0$

$\therefore r^n \vec{r}$ is irrotational //

Laplacian operator :- It is denoted by " ∇^2 " and is defined

as $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. If ϕ is any scalar point function.

consider $\text{div grad } \phi = \nabla \cdot \nabla \phi$

$$= \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \cdot \left(\vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z} \right)$$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \phi$$

$$= \nabla^2 \phi$$

$$\therefore \nabla^2 \phi = \nabla \cdot \nabla \phi$$

1. Prove that $\nabla^2 (f(r)) = \frac{d^2 f}{dr^2} + \frac{2}{r} \frac{df}{dr}$ where $r = |\vec{r}|$

Sol:- Proof:- Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$r^2 = x^2 + y^2 + z^2 \rightarrow (1)$$

diff eqn (1) partially w.r.t 'x'

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

consider $\nabla^2(f(r)) = \nabla \cdot \nabla(f(r))$

$$\text{Now } \nabla(f(r)) = \sum \bar{i} \frac{\partial}{\partial x} f(r)$$

$$= \sum \bar{i} f'(r) \frac{\partial r}{\partial x}$$

$$= \sum \bar{i} f'(r) \frac{x}{r}$$

$$\text{then } \nabla \cdot \nabla(f(r)) = \sum \frac{\partial}{\partial x} \left(\frac{f'(r)x}{r} \right)$$

$$= \sum \left[\frac{r \left[f'(r) + x f''(r) \frac{\partial r}{\partial x} \right] - f'(r)x \frac{\partial r}{\partial x}}{r^2} \right]$$

$$= \sum \left[\frac{r \left(f'(r) + x f''(r) \frac{x}{r} \right)}{r^2} - \frac{f'(r)x \cdot \frac{x}{r}}{r^2} \right]$$

$$= \sum \frac{f'(r)}{r} + \sum \frac{f''(r)}{r^2} x^2 - \sum \frac{f'(r)}{r^3} x^2$$

$$= \frac{3f'(r)}{r} + \frac{f''(r)}{r^2} \sum x^2 - \frac{f'(r)}{r^3} \sum x^2$$

$$= \frac{3f'(r)}{r} + \frac{f''(r)}{r^2} (x^2 + y^2 + z^2) - \frac{f'(r)}{r^3} (x^2 + y^2 + z^2)$$

$$= \frac{3f'(r)}{r} + \frac{f''(r)}{r^2} (r^2) - \frac{f'(r)}{r^3} (r^2)$$

$$= \frac{3f'(r)}{r} + f''(r) - \frac{f'(r)}{r}$$

$$\nabla \cdot \nabla(f(r)) = f''(r) + \frac{2}{r} f'(r)$$

2. Evaluate the following

(i) $\nabla^2(\log r)$

(ii) $\nabla^2(r^n)$

(iii) $\nabla^2\left(\frac{1}{r}\right)$

3. If $f = (x^2 + y^2 + z^2)^{-n}$ then find $\text{div}(\text{grad } f)$ and determine 'n' if $\text{div}(\text{grad } f) = 0$.

Sol:- Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$|\vec{r}| = r = \sqrt{x^2 + y^2 + z^2}$$

$$r^2 = x^2 + y^2 + z^2 \rightarrow \textcircled{1}$$

$$\text{Let } f = (x^2 + y^2 + z^2)^{-n} = (r^2)^{-n} \\ = r^{-2n} = f(r)$$

$$f(r) = r^{-2n}$$

$$f'(r) = -2n r^{-2n-1}$$

$$f''(r) = (-2n)(-2n-1) r^{-2n-2}$$

$$\text{Now } \text{div}(\text{grad } f) = \nabla \cdot \nabla f \\ = \nabla^2 f$$

$$= f''(r) + \frac{2}{r} f'(r)$$

$$= (-2n)(-2n-1) r^{-2n-2} + \frac{2}{r} (-2n) r^{-2n-1}$$

$$= 2n(2n+1) r^{-2n-2} - 4n r^{-2n-2}$$

$$= r^{-2n-2} (2n(2n+1) - 4n)$$

$$= r^{-2n-2} (4n^2 + 2n - 4n)$$

$$= r^{-2n-2} (4n^2 - 2n)$$

$$= r^{-2n-2} \cdot 2n(2n-1)$$

$$\therefore \text{div}(\text{grad } f) = 0$$

$$\Rightarrow r^{-2n-2} \cdot 2n(2n-1) = 0$$

$$\Rightarrow 2n(2n-1) = 0$$

$$\Rightarrow 2n = 0, \quad 2n-1 = 0$$

$$n = 0, \quad n = \frac{1}{2}$$

$$\therefore n = 0, \frac{1}{2}$$

2.(*) $\nabla^2(\log r)$

Proof:- Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$|\vec{r}| = r = \sqrt{x^2 + y^2 + z^2}$$

$$r^2 = x^2 + y^2 + z^2 \rightarrow \textcircled{1}$$

diff eqn ① partially w.r.t 'x'

$$2r \frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

consider $\nabla^2(\log r) = \nabla \cdot \nabla(\log r)$

Now $\nabla(\log r) = \sum \hat{i} \frac{\partial}{\partial x}(\log r)$

$$= \sum \hat{i} \frac{1}{r} \frac{\partial r}{\partial x}$$

$$= \sum \hat{i} \frac{1}{r} \frac{x}{r}$$

$$= \sum \hat{i} \frac{1}{r^2} x$$

then $\nabla \cdot \nabla(\log r) = \sum \frac{\partial}{\partial x} \left(\frac{x}{r^2} \right)$

$$= \sum \left(\frac{r^2(1) - x \cdot 2r \frac{\partial r}{\partial x}}{r^4} \right)$$

$$= \sum \left(\frac{r^2}{r^4} - \frac{2r \cdot x \cdot \frac{x}{r}}{r^4} \right)$$

$$= \sum \left(\frac{1}{r^2} - \frac{2x^2}{r^4} \right)$$

$$= \sum \frac{1}{r^2} - \sum \frac{2x^2}{r^4}$$

$$= \frac{3}{r^2} - \frac{2}{r^4} \sum x^2$$

$$= \frac{3}{r^2} - \frac{2}{r^4} (x^2 + y^2 + z^2)$$

$$= \frac{3}{r^2} - \frac{2}{r^4} (r^2)$$

$$= \frac{3}{r^2} - \frac{2}{r^2}$$

$$\nabla \cdot \nabla(\log r) = \frac{1}{r^2} //$$

(ii) $\nabla^2(r^n)$.

Sol - proof :- Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$|\vec{r}| = r = \sqrt{x^2 + y^2 + z^2}$$

$$r^2 = x^2 + y^2 + z^2 \rightarrow (1)$$

diff. eqn (1) w.r.t 'x'

$$2r \frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

consider $\nabla^2(r^n) = \nabla \cdot \nabla(r^n)$

$$= \sum \bar{i} \frac{\partial}{\partial x} (r^n)$$

$$= \sum \bar{i} n r^{n-1} \frac{\partial r}{\partial x}$$

$$= \sum \bar{i} n r^{n-1} \frac{x}{r}$$

then $\nabla \cdot \nabla(r^n) = \sum \frac{\partial}{\partial x} \left(\frac{n r^{n-1} x}{r} \right)$

$$= \sum n \frac{\partial}{\partial x} \frac{r^{n-1} x}{r}$$

$$= \sum n \left[\frac{r(x(n-1)r^{n-2} \frac{\partial r}{\partial x} + r^{n-1}) - x r^{n-1} \frac{\partial r}{\partial x}}{r^2} \right]$$

$$= \sum n \left[\frac{r}{r^2} \left(x(n-1)r^{n-2} \frac{x}{r} + r^{n-1} \right) - \frac{x r^{n-1} \frac{x}{r}}{r^2} \right]$$

$$= \sum n \left[\frac{(n-1)r^{n-2} x^2}{r^2} + \frac{r^{n-1}}{r} - \frac{r^{n-1} x^2}{r^3} \right]$$

$$= \sum \frac{n(n-1)r^{n-2} x^2}{r^2} + \sum \frac{n r^{n-1}}{r} - \sum \frac{n r^{n-1} x^2}{r^3}$$

$$= \frac{n(n-1)r^{n-2}}{r^2} \sum x^2 + \frac{3nr^{n-1}}{r} - \frac{nr^{n-1}}{r^3} \sum x^2$$

$$= \frac{n(n-1)r^{n-2}}{r^2} (x^2 + y^2 + z^2) + \frac{3nr^{n-1}}{r} - \frac{nr^{n-1}}{r^3} (x^2 + y^2 + z^2)$$

$$= \frac{n(n-1)r^{n-2}}{r^2} (x^2) + \frac{3nr^{n-1}}{r} - \frac{nr^{n-1}}{r^3} (x^2)$$

$$= n(n-1)r^{n-2} + \frac{3nr^{n-1}}{r} - \frac{nr^{n-1}}{r}$$

$$\nabla \cdot \nabla r^n = n(n-1)r^{n-2} + \frac{3}{r} nr^{n-1}$$

(iii) $\nabla^2 \left(\frac{1}{r} \right)$

Sol:- proof :- let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$
 $|\vec{r}| = r = \sqrt{x^2 + y^2 + z^2}$

$$r^2 = x^2 + y^2 + z^2 \rightarrow (1)$$

diff. eqn (1) w.r.t 'x' partially.

$$2r \frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly $\frac{\partial^2 r}{\partial x^2} = \frac{-y}{r^3}$ $\frac{\partial r}{\partial z} = \frac{z}{r^3}$

$$\nabla^2 \left(\frac{1}{r} \right) = \nabla \cdot \nabla \left(\frac{1}{r} \right)$$

$$\text{Now } \nabla \left(\frac{1}{r} \right) = \sum \vec{i} \frac{\partial}{\partial x} \left(\frac{1}{r} \right)$$

$$= \sum \vec{i} \frac{-1}{r^2} \frac{\partial r}{\partial x}$$

$$= \sum \vec{i} \frac{-1}{r^2} \frac{x}{r}$$

$$= \sum \vec{i} \frac{-x}{r^3}$$

$$\nabla \cdot \nabla \left(\frac{1}{r} \right) = \sum \frac{\partial}{\partial x} \left(\frac{-x}{r^3} \right)$$

$$= \sum \frac{r^3(-1) - (-x)3r^2 \frac{\partial r}{\partial x}}{(r^3)^2}$$

$$= \sum \frac{-r^3 + x3r^2 \frac{x}{r}}{r^6}$$

$$= \sum \frac{-r^3}{r^6} + \sum \frac{3x^2}{r^5}$$

$$= \frac{-3r^3}{r^6} + \frac{3}{r^5} \sum x^2$$

$$= \frac{-3}{r^3} + \frac{3}{r^5} (x^2 + y^2 + z^2)$$

$$= \frac{-3}{r^3} + \frac{3}{r^5} (r^2)$$

$$= \frac{-3}{r^3} + \frac{3}{r^3}$$

$$\nabla \cdot \nabla \left(\frac{1}{r} \right) = 0 //$$

(ii)

Sc

Important formula's

$$* \text{grad } \phi = \nabla \phi = \bar{i} \frac{\partial \phi}{\partial x} + \bar{j} \frac{\partial \phi}{\partial y} + \bar{k} \frac{\partial \phi}{\partial z} = \sum \bar{i} \frac{\partial \phi}{\partial x}$$

$$* \text{div } \vec{f} = \nabla \cdot \vec{f} = \bar{i} \cdot \frac{\partial \vec{f}}{\partial x} + \bar{j} \cdot \frac{\partial \vec{f}}{\partial y} + \bar{k} \cdot \frac{\partial \vec{f}}{\partial z} = \sum \bar{i} \cdot \frac{\partial \vec{f}}{\partial x}$$

$$* (\vec{A} \cdot \nabla) \phi = (\vec{A} \cdot \bar{i}) \frac{\partial \phi}{\partial x} + (\vec{A} \cdot \bar{j}) \frac{\partial \phi}{\partial y} + (\vec{A} \cdot \bar{k}) \frac{\partial \phi}{\partial z} = \sum (\vec{A} \cdot \bar{i}) \frac{\partial \phi}{\partial x}$$

$$* (\vec{A} \cdot \nabla) \vec{f} = \sum (\vec{A} \cdot \bar{i}) \frac{\partial \vec{f}}{\partial x}$$

$$* \text{Curl } \vec{f} = \nabla \times \vec{f} = \bar{i} \times \frac{\partial \vec{f}}{\partial x} + \bar{j} \times \frac{\partial \vec{f}}{\partial y} + \bar{k} \times \frac{\partial \vec{f}}{\partial z} = \sum \bar{i} \times \frac{\partial \vec{f}}{\partial x}$$

$$* (\vec{A} \times \nabla) \phi = (\vec{A} \times \bar{i}) \frac{\partial \phi}{\partial x} + (\vec{A} \times \bar{j}) \frac{\partial \phi}{\partial y} + (\vec{A} \times \bar{k}) \frac{\partial \phi}{\partial z} = \sum (\vec{A} \times \bar{i}) \frac{\partial \phi}{\partial x}$$

$$* \nabla^2 \phi = \nabla \cdot \nabla \phi$$

* If $\bar{i}, \bar{j}, \bar{k}$ are three mutually perpendicular unit vector

$$\text{then } \bar{i} \cdot \bar{i} = \bar{j} \cdot \bar{j} = \bar{k} \cdot \bar{k} = 1$$

$$\bar{i} \cdot \bar{j} = \bar{j} \cdot \bar{k} = \bar{k} \cdot \bar{i} = 0$$

$$\bar{i} \times \bar{i} = \bar{j} \times \bar{j} = \bar{k} \times \bar{k} = 0$$

$$\bar{i} \times \bar{j} = \bar{k}, \quad \bar{j} \times \bar{k} = \bar{i}, \quad \bar{k} \times \bar{i} = \bar{j}$$

* $\vec{a} = a_1 \bar{i} + a_2 \bar{j} + a_3 \bar{k}$, $\vec{b} = b_1 \bar{i} + b_2 \bar{j} + b_3 \bar{k}$ then

$$(i) \vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$(ii) \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$

$$(iii) \vec{a} \cdot \vec{a} = a_1^2 + a_2^2 + a_3^2 = |\vec{a}|^2$$

* If $\vec{a}, \vec{b}$ are any two vectors and they are $\perp$ to each other then $\vec{a} \cdot \vec{b} = 0$

* Let θ be the angle between two vectors $\vec{a}$ and $\vec{b}$ then

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|}$$

* If $\vec{a} = a_1 \bar{i} + a_2 \bar{j} + a_3 \bar{k}$, $\vec{b} = b_1 \bar{i} + b_2 \bar{j} + b_3 \bar{k}$ then

$$(i) \vec{a} \times \vec{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$(ii) \vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$$

* If $\vec{a} = a_1 \bar{i} + a_2 \bar{j} + a_3 \bar{k}$, $\vec{b} = b_1 \bar{i} + b_2 \bar{j} + b_3 \bar{k}$, $\vec{c} = c_1 \bar{i} + c_2 \bar{j} + c_3 \bar{k}$ then the scalar triple product (or) box product.

$$[\vec{a}, \vec{b}, \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

1. If $a = x+y+z$, $b = x^2+y^2+z^2$, $c = xy+yz+zx$ prove that $[\text{grad } a, \text{grad } b, \text{grad } c] = 0$

Sol:- Given $a = x+y+z$

$$\text{grad } a = \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z}$$

* If $\vec{a}, \vec{b}, \vec{c}$ are any three vectors

$$(i) [\vec{a}, \vec{b}, \vec{c}] = [\vec{b}, \vec{c}, \vec{a}] = [\vec{c}, \vec{a}, \vec{b}]$$

$$(ii) [\vec{a}, \vec{a}, \vec{b}] = 0$$

$$(iii) [\vec{a}, \vec{b}, \vec{c}] = \vec{a} \cdot (\vec{b} \times \vec{c})$$

$$(iv) \vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{a} \cdot (\vec{c} \times \vec{b}) \cdot (-1)$$

$$(v) \vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c}$$

$$(vi) (\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{b} \cdot \vec{c}) \vec{a}$$

* If $\vec{a}, \vec{b}$ are any two vectors (i) $\frac{\partial}{\partial x} (\vec{a} \pm \vec{b}) = \frac{\partial \vec{a}}{\partial x} \pm \frac{\partial \vec{b}}{\partial x}$

$$(ii) \frac{\partial}{\partial x} (\vec{a} \cdot \vec{b}) = \frac{\partial \vec{a}}{\partial x} \cdot \vec{b} + \vec{a} \cdot \frac{\partial \vec{b}}{\partial x}$$

$$(iii) \frac{\partial}{\partial x} (\vec{a} \times \vec{b}) = \frac{\partial \vec{a}}{\partial x} \times \vec{b} + \vec{a} \times \frac{\partial \vec{b}}{\partial x}$$

* If $\vec{a}$ and ϕ is a vector and scalar point functions then

$$\frac{\partial}{\partial x} (\phi \vec{a}) = \frac{\partial \phi}{\partial x} \vec{a} + \phi \frac{\partial \vec{a}}{\partial x}$$

$$* \frac{\partial}{\partial x} (k \vec{a}) = k \frac{\partial \vec{a}}{\partial x}$$

(i)

Sx

Vector Identities :-

1. Prove that $\text{curl grad } \phi = 0$ where ϕ is any scalar point function.

Sol:- Consider $\text{grad } \phi = \bar{i} \frac{\partial \phi}{\partial x} + \bar{j} \frac{\partial \phi}{\partial y} + \bar{k} \frac{\partial \phi}{\partial z}$

$$\text{curl grad } \phi = \nabla \times (\text{grad } \phi) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \bar{i} \left[\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y} \right] - \bar{j} \left[\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial z \partial x} \right] + \bar{k} \left[\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x} \right]$$

$$\text{curl grad } \phi = 0$$

$\Rightarrow$ grad ϕ is always irrotational //

2. P.T $\text{div curl } \vec{f} = 0$ where $\vec{f}$ is any vector point function.

Sol:- Let $\vec{f} = f_1 \bar{i} + f_2 \bar{j} + f_3 \bar{k}$ is a vector point function.

$$\text{curl } \vec{f} = \nabla \times \vec{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \bar{i} \left[\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right] + \bar{j} \left[\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right] + \bar{k} \left[\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right]$$

$$\begin{aligned} \text{div curl } \vec{f} &= \frac{\partial}{\partial x} \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) - \frac{\partial}{\partial y} \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right) + \frac{\partial}{\partial z} \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \\ &= \frac{\partial^2 f_3}{\partial x \partial y} - \frac{\partial^2 f_2}{\partial x \partial z} - \frac{\partial^2 f_1}{\partial y \partial z} + \frac{\partial^2 f_3}{\partial y \partial x} + \frac{\partial^2 f_1}{\partial z \partial y} - \frac{\partial^2 f_2}{\partial z \partial x} \end{aligned}$$

$$\text{div curl } \vec{f} = 0$$

$\Rightarrow$ curl $\vec{f}$ is always solenoidal. //

3. If f & g are two scalar point functions, P.T

$$\text{div}(f \nabla g) = f \nabla^2 g + \nabla f \cdot \nabla g$$

Sol:- Consider $\nabla g = \bar{i} \frac{\partial g}{\partial x} + \bar{j} \frac{\partial g}{\partial y} + \bar{k} \frac{\partial g}{\partial z}$

$$\nabla f = \bar{i} \frac{\partial f}{\partial x} + \bar{j} \frac{\partial f}{\partial y} + \bar{k} \frac{\partial f}{\partial z}$$

$$\nabla f \cdot \nabla g = \left(\bar{i} \frac{\partial f}{\partial x} + \bar{j} \frac{\partial f}{\partial y} + \bar{k} \frac{\partial f}{\partial z} \right) \cdot \left(\bar{i} \frac{\partial g}{\partial x} + \bar{j} \frac{\partial g}{\partial y} + \bar{k} \frac{\partial g}{\partial z} \right)$$

$$= \frac{\partial g}{\partial x} \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \frac{\partial f}{\partial y} + \frac{\partial g}{\partial z} \frac{\partial f}{\partial z} \rightarrow (1)$$

$$\text{Now } f \nabla g = \bar{i} f \frac{\partial g}{\partial x} + \bar{j} f \frac{\partial g}{\partial y} + \bar{k} f \frac{\partial g}{\partial z}$$

$$\text{div}(f \nabla g) = \nabla \cdot (f \nabla g) = \left(\bar{i} \frac{\partial}{\partial x} + \bar{j} \frac{\partial}{\partial y} + \bar{k} \frac{\partial}{\partial z} \right) \cdot \left(\bar{i} f \frac{\partial g}{\partial x} + \bar{j} f \frac{\partial g}{\partial y} + \bar{k} f \frac{\partial g}{\partial z} \right)$$

$$\begin{aligned} &= \frac{\partial}{\partial x} \left(f \frac{\partial g}{\partial x} \right) + \frac{\partial}{\partial y} \left(f \frac{\partial g}{\partial y} \right) + \frac{\partial}{\partial z} \left(f \frac{\partial g}{\partial z} \right) \\ &= f \frac{\partial^2 g}{\partial x^2} + \frac{\partial g}{\partial x} \frac{\partial f}{\partial x} + f \frac{\partial^2 g}{\partial y^2} + \frac{\partial g}{\partial y} \frac{\partial f}{\partial y} + f \frac{\partial^2 g}{\partial z^2} + \frac{\partial g}{\partial z} \frac{\partial f}{\partial z} \\ &= f \left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2} \right) + \frac{\partial g}{\partial x} \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \frac{\partial f}{\partial y} + \frac{\partial g}{\partial z} \frac{\partial f}{\partial z} \end{aligned}$$

$$\text{div}(f \nabla g) = f \nabla^2 g + \nabla f \cdot \nabla g //$$

4. If $\bar{a}$ & ϕ are differentiable vector and scalar point funct

then p.T (i) $\text{div}(\phi \bar{a}) = (\text{grad } \phi) \cdot \bar{a} + \phi \text{div } \bar{a}$

(ii) $\text{curl}(\phi \bar{a}) = (\text{grad } \phi) \times \bar{a} + \phi \text{curl } \bar{a}$

Sol:- (i) consider $\text{div}(\phi \bar{a}) = \nabla \cdot \phi \bar{a}$

$$= \sum \bar{i} \cdot \frac{\partial}{\partial x} (\phi \bar{a})$$

$$= \sum \bar{i} \cdot \left(\frac{\partial \phi}{\partial x} \bar{a} + \phi \frac{\partial \bar{a}}{\partial x} \right)$$

$$= \sum \bar{i} \cdot \frac{\partial \phi}{\partial x} \bar{a} + \sum \bar{i} \cdot \left(\phi \frac{\partial \bar{a}}{\partial x} \right)$$

$$= \sum \bar{i} \frac{\partial \phi}{\partial x} \cdot \bar{a} + \phi \sum \bar{i} \cdot \frac{\partial \bar{a}}{\partial x}$$

$$= \nabla \phi \cdot \bar{a} + \phi (\nabla \cdot \bar{a})$$

$$\text{div}(\phi \bar{a}) = (\text{grad } \phi) \cdot \bar{a} + \phi (\text{div } \bar{a}) //$$

(ii) consider $\text{curl}(\phi \bar{a}) = \nabla \times \phi \bar{a}$

$$= \sum \bar{i} \times \frac{\partial}{\partial x} (\phi \bar{a})$$

$$= \sum \bar{i} \times \left(\frac{\partial \phi}{\partial x} \bar{a} + \phi \frac{\partial \bar{a}}{\partial x} \right)$$

$$= \sum \bar{i} \times \frac{\partial \phi}{\partial x} \bar{a} + \sum \bar{i} \times \left(\phi \frac{\partial \bar{a}}{\partial x} \right)$$

$$= \sum \bar{i} \frac{\partial \phi}{\partial x} \times \bar{a} + \phi \sum \bar{i} \times \frac{\partial \bar{a}}{\partial x}$$

$$= \nabla \phi \times \bar{a} + \phi (\nabla \times \bar{a})$$

$$= (\text{grad } \phi) \times \bar{a} + \phi \text{curl } \bar{a}$$

$$\text{curl}(\phi \bar{a}) = \text{curl}(\text{grad } \phi) \times \bar{a} + \phi \text{curl } \bar{a}$$

5. Prove that $\nabla \cdot (\bar{A} \times \bar{B}) = \bar{B} \cdot (\nabla \times \bar{A}) - \bar{A} \cdot (\nabla \times \bar{B})$

Sol:-

(or)

$$\nabla \cdot (\bar{A} \times \bar{B}) = \bar{B} \cdot \text{curl } \bar{A} - \bar{A} \cdot \text{curl } \bar{B}$$

Sol:- proof: consider

$$\nabla \cdot (\bar{A} \times \bar{B}) = \sum \bar{i} \cdot \frac{\partial}{\partial x} (\bar{A} \times \bar{B})$$

$$= \sum \bar{i} \cdot \left(\frac{\partial \bar{A}}{\partial x} \times \bar{B} + \bar{A} \times \frac{\partial \bar{B}}{\partial x} \right)$$

$$= \sum \bar{i} \cdot \left(\frac{\partial \bar{A}}{\partial x} \times \bar{B} \right) + \sum \bar{i} \cdot \left(\bar{A} \times \frac{\partial \bar{B}}{\partial x} \right)$$

$$= \sum \left(\bar{i} \times \frac{\partial \bar{A}}{\partial x} \right) \cdot \bar{B} - \sum \bar{i} \cdot \left(\frac{\partial \bar{B}}{\partial x} \times \bar{A} \right) \quad \because \bar{a} \cdot (\bar{b} \times \bar{c}) = (\bar{a} \times \bar{b}) \cdot \bar{c}$$

$$= \sum \bar{i} \times \bar{B} \cdot \sum \frac{\partial \bar{A}}{\partial x} - \sum \left(\bar{i} \times \frac{\partial \bar{B}}{\partial x} \right) \cdot \bar{A}$$

$$= (\nabla \times \bar{A}) \cdot \bar{B} - (\nabla \times \bar{B}) \cdot \bar{A}$$

$$= \bar{B} \cdot (\nabla \times \bar{A}) - \bar{A} \cdot (\nabla \times \bar{B})$$

$$\therefore \nabla \cdot (\bar{A} \times \bar{B}) = \bar{B} \cdot (\nabla \times \bar{A}) - \bar{A} \cdot (\nabla \times \bar{B}) //$$

6. Prove that $\nabla \times (\bar{A} \times \bar{B}) = (\bar{B} \cdot \nabla) \bar{A} - \bar{B} (\nabla \cdot \bar{A}) - (\bar{A} \cdot \nabla) \bar{B} + (\nabla \cdot \bar{B}) \bar{A}$

Sol:- consider $\nabla \times (\bar{A} \times \bar{B}) = \sum \bar{i} \times \frac{\partial}{\partial x} (\bar{A} \times \bar{B}) \quad \left(\nabla \times \bar{f} = \sum \bar{i} \times \frac{\partial \bar{f}}{\partial x} \right)$

$$= \sum \bar{i} \times \left(\frac{\partial \bar{A}}{\partial x} \times \bar{B} + \bar{A} \times \frac{\partial \bar{B}}{\partial x} \right)$$

$$= \sum \bar{i} \times \left(\frac{\partial \bar{A}}{\partial x} \times \bar{B} \right) + \sum \bar{i} \times \left(\bar{A} \times \frac{\partial \bar{B}}{\partial x} \right)$$

$$= \sum \left[\left(\bar{i} \cdot \bar{B} \right) \frac{\partial \bar{A}}{\partial x} - \left(\bar{i} \cdot \frac{\partial \bar{A}}{\partial x} \right) \bar{B} \right] + \sum \left[\left(\bar{i} \cdot \frac{\partial \bar{B}}{\partial x} \right) \bar{A} - \left(\bar{i} \cdot \bar{A} \right) \frac{\partial \bar{B}}{\partial x} \right]$$

$$\begin{aligned} \therefore (\bar{A} \cdot \nabla) \bar{B} &= (\bar{A} \cdot \bar{i}) \frac{\partial \bar{B}}{\partial x} + (\bar{A} \cdot \bar{j}) \frac{\partial \bar{B}}{\partial y} + (\bar{A} \cdot \bar{k}) \frac{\partial \bar{B}}{\partial z} \\ &= \sum (\bar{A} \cdot \bar{i}) \frac{\partial \bar{B}}{\partial x} \\ &= \sum (\bar{B} \cdot \bar{i}) \frac{\partial \bar{A}}{\partial x} - (\nabla \cdot \bar{A}) \bar{B} + (\nabla \cdot \bar{B}) \bar{A} - \sum (\bar{A} \cdot \bar{i}) \frac{\partial \bar{B}}{\partial x} \end{aligned}$$

$$\nabla \times (\bar{A} \times \bar{B}) = (\bar{B} \cdot \nabla) \bar{A} - \bar{B} (\nabla \cdot \bar{A}) + (\nabla \cdot \bar{B}) \bar{A} - (\bar{A} \cdot \nabla) \bar{B} //$$

$$7. \text{ P.T } \nabla(\vec{A} \cdot \vec{B}) = (\vec{B} \cdot \nabla)\vec{A} + (\vec{A} \cdot \nabla)\vec{B} + \vec{B} \times (\nabla \times \vec{A}) + \vec{A} \times (\nabla \times \vec{B})$$

Sol:- Consider $\vec{A} \times (\nabla \times \vec{B}) = \vec{A} \times \left(\sum \vec{i} \times \frac{\partial \vec{B}}{\partial x} \right)$

$$= \sum \left(\vec{A} \times \left(\vec{i} \times \frac{\partial \vec{B}}{\partial x} \right) \right)$$

$$= \sum \left(\left(\vec{A} \cdot \frac{\partial \vec{B}}{\partial x} \right) \vec{i} - (\vec{A} \cdot \vec{i}) \frac{\partial \vec{B}}{\partial x} \right)$$

$$= \sum \vec{i} \left(\vec{A} \cdot \frac{\partial \vec{B}}{\partial x} \right) - \sum (\vec{A} \cdot \vec{i}) \frac{\partial \vec{B}}{\partial x}$$

$$= \sum \vec{i} \left(\vec{A} \cdot \frac{\partial \vec{B}}{\partial x} \right) - \sum (\vec{A} \cdot \nabla) \vec{B} \rightarrow (1)$$

Similarly

$$\vec{B} \times (\nabla \times \vec{A}) = \sum \vec{i} \left(\vec{B} \cdot \frac{\partial \vec{A}}{\partial x} \right) - (\vec{B} \cdot \nabla) \vec{A} \rightarrow (2)$$

$$(1) + (2) \Rightarrow$$

$$\vec{A} \times (\nabla \times \vec{B}) + \vec{B} \times (\nabla \times \vec{A}) = \sum \vec{i} \left(\vec{A} \cdot \frac{\partial \vec{B}}{\partial x} \right) - (\vec{A} \cdot \nabla) \vec{B} + \sum \vec{i} \left(\vec{B} \cdot \frac{\partial \vec{A}}{\partial x} \right) - (\vec{B} \cdot \nabla) \vec{A}$$

$$\Rightarrow \vec{A} \times (\nabla \times \vec{B}) + \vec{B} \times (\nabla \times \vec{A}) + (\vec{A} \cdot \nabla) \vec{B} + (\vec{B} \cdot \nabla) \vec{A} = \sum \vec{i} \left(\vec{A} \cdot \frac{\partial \vec{B}}{\partial x} + \vec{B} \cdot \frac{\partial \vec{A}}{\partial x} \right)$$

$$= \sum \vec{i} \frac{\partial (\vec{A} \cdot \vec{B})}{\partial x}$$

$$= \sum \vec{i} \frac{\partial (\vec{A} \cdot \vec{B})}{\partial x}$$

$$= \nabla (\vec{A} \cdot \vec{B}) //$$

$$8. \text{ P.T } \nabla \times (\nabla \times \vec{A}) = \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

Sol:- Proof:- Let $\vec{A} = A_1 \vec{i} + A_2 \vec{j} + A_3 \vec{k}$ be a vector point function

then $\nabla \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$

$$= \vec{i} \left(\frac{\partial}{\partial y} A_3 - \frac{\partial A_2}{\partial z} \right) - \vec{j} \left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z} \right) + \vec{k} \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right)$$

$$\nabla \times (\nabla \times \vec{A}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} & - \left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z} \right) & \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \end{vmatrix}$$

$$= \sum \vec{i} \left(\frac{\partial}{\partial y} \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) + \frac{\partial}{\partial z} \left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z} \right) \right)$$

$$\begin{aligned}
&= \sum \bar{i} \left[\frac{\partial^2 A_2}{\partial y \partial x} - \frac{\partial^2 A_1}{\partial y^2} + \frac{\partial^2 A_3}{\partial z \partial x} - \frac{\partial^2 A_1}{\partial z^2} \right] \\
&= \sum \bar{i} \left[\frac{\partial^2 A_2}{\partial x \partial y} + \frac{\partial^2 A_3}{\partial x \partial z} - \frac{\partial^2 A_1}{\partial y^2} - \frac{\partial^2 A_1}{\partial z^2} \right] \\
&= \sum \bar{i} \left[\frac{\partial^2 A_1}{\partial x^2} + \frac{\partial^2 A_2}{\partial x \partial y} + \frac{\partial^2 A_3}{\partial x \partial z} - \frac{\partial^2 A_1}{\partial x^2} - \frac{\partial^2 A_1}{\partial y^2} - \frac{\partial^2 A_1}{\partial z^2} \right] \\
&= \sum \bar{i} \left[\frac{\partial}{\partial x} \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right) - \frac{\partial}{\partial x} \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_1}{\partial y} + \frac{\partial A_1}{\partial z} \right) \right] \\
&= \sum \bar{i} \left[\frac{\partial}{\partial x} (\nabla \cdot \vec{A}) - \nabla^2 A_1 \right] \\
&= \sum \bar{i} \frac{\partial}{\partial x} (\nabla \cdot \vec{A}) - \nabla^2 \sum A_1 \bar{i} \\
&= \nabla (\nabla \cdot \vec{A}) - \nabla^2 (A_1 \bar{i} + A_2 \bar{j} + A_3 \bar{k})
\end{aligned}$$

$$\nabla \times (\nabla \times \vec{A}) = \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A} \quad //$$

Q. If $\vec{f}$ is a solenoidal vector, p.T $\text{curl curl curl curl } \vec{f} = \vec{f}$

Sol:- Since $\vec{f}$ is solenoidal vector then

$$\text{div } \vec{f} = 0$$

$$\nabla \cdot \vec{f} = 0 \rightarrow (1)$$

$$\begin{aligned}
\text{Consider } \text{curl curl } \vec{f} &= \nabla \times (\nabla \times \vec{f}) \\
&= \nabla (\nabla \cdot \vec{f}) - \nabla^2 \vec{f} \\
&= \nabla (0) - \nabla^2 \vec{f} \\
&= -\nabla^2 \vec{f} = -\vec{a}
\end{aligned}$$

$$\begin{aligned}
\therefore \text{curl curl curl curl } \vec{f} &= \text{curl curl } (-\vec{a}) \\
&= \nabla \times \nabla \times (-\vec{a}) \\
&= -(\nabla \times \nabla \times \vec{a}) \\
&= -(\nabla (\nabla \cdot \vec{a}) - \nabla^2 \vec{a}) \\
&= -\nabla (\nabla \cdot \vec{a}) + \nabla^2 \vec{a} \\
&= -\nabla (\nabla \cdot \nabla^2 \vec{f}) + \nabla^2 \nabla^2 \vec{f} \\
&= -\nabla (\nabla \cdot \nabla^2 \vec{f}) + \nabla^4 \vec{f} \rightarrow (2)
\end{aligned}$$

$$\begin{aligned}
\text{Consider } \nabla^2 \vec{f} &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) (f_1 \bar{i} + f_2 \bar{j} + f_3 \bar{k}) \\
&= \sum \bar{i} \left(\frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_1}{\partial y^2} + \frac{\partial^2 f_1}{\partial z^2} \right)
\end{aligned}$$

$$= \sum \bar{i} \nabla^2 f_1$$

$$\nabla^2 \bar{f} = \bar{i} \nabla^2 f_1 + \bar{j} \nabla^2 f_2 + \bar{k} \nabla^2 f_3$$

$$\text{Now } \nabla \cdot \nabla^2 \bar{f} = \left[\bar{i} \frac{\partial}{\partial x} + \bar{j} \frac{\partial}{\partial y} + \bar{k} \frac{\partial}{\partial z} \right] \cdot \left[\bar{i} \nabla^2 f_1 + \bar{j} \nabla^2 f_2 + \bar{k} \nabla^2 f_3 \right]$$

$$= \nabla^2 \frac{\partial f_1}{\partial x} + \nabla^2 \frac{\partial f_2}{\partial y} + \nabla^2 \frac{\partial f_3}{\partial z}$$

$$= \nabla^2 \left[\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \right]$$

$$= \nabla^2 (\nabla \cdot \bar{f})$$

$$= \nabla^2 (0)$$

$$\nabla \cdot \nabla^2 \bar{f} = 0$$

from (5)

$$\text{curl curl curl curl } \bar{f} = -\nabla(0) + \nabla^4 \bar{f}$$

$$= \nabla^4 \bar{f} //$$

10. If ϕ satisfies Laplace's equation then show that $\nabla \phi$ is both solenoidal and irrotational.

Sol:- Given the function ' ϕ ' satisfies Laplace's eqn

$$\text{i.e.; } \nabla^2 \phi = 0$$

$$\nabla \cdot \nabla \phi = 0$$

$$\text{div}(\text{grad } \phi) = 0$$

$\Rightarrow$ grad ϕ is Solenoidal

$$\text{w.k.t } \text{curl}(\text{grad } \phi) = 0$$

$\Rightarrow$ grad ϕ is irrotational.

$\therefore$ grad ϕ is Solenoidal & irrotational //

Vector Integration

Let $\vec{g}(t)$ is a differentiable vector function in variable 't' such that $\frac{d}{dt} \vec{g}(t) = \vec{f}(t)$ then $\vec{g}(t)$ is called the integral of $\vec{f}(t)$ and is denoted by

$$\int \vec{f}(t) dt = \vec{g}(t) + \vec{c}, \text{ where } \vec{c} \text{ is a constant vector.}$$

Properties :-

$$* \int k \vec{f}(t) dt = k \int \vec{f}(t) dt$$

$$* \int (\vec{f}(t) \pm \vec{g}(t)) dt = \int \vec{f}(t) dt \pm \int \vec{g}(t) dt$$

$$* \vec{f}(t) = f_1(t)\vec{i} + f_2(t)\vec{j} + f_3(t)\vec{k}$$

$$\int \vec{f}(t) dt = \vec{i} \int f_1(t) dt + \vec{j} \int f_2(t) dt + \vec{k} \int f_3(t) dt$$

$$* \int_a^b \vec{f}(t) dt = [\vec{g}(t)]_a^b = \vec{g}(b) - \vec{g}(a) \text{ is called definite integral of } \vec{f}(t).$$

Line integral :-

For the ordinary definite integral $\int_a^b f(x) dx$, the region of integration is an interval $a \leq x \leq b$ on the x-axis that is we integrate along the x-axis. This concept can be generalized to definite define a definite integral along a curve.

The line integral of a vector point function ' $\vec{F}$ ' over a curve 'C' from the point A to B is defined as $\int_C \vec{F} \cdot d\vec{r} = \int_A^B \vec{F} \cdot d\vec{r}$

$$* \text{If } \vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$$

$$\text{let } \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\text{then } d\vec{r} = d(x\vec{i} + y\vec{j} + z\vec{k}) \\ = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\therefore \vec{F} \cdot d\vec{r} = F_1 dx + F_2 dy + F_3 dz$$

* If $x = x(t)$, $y = y(t)$, $z = z(t)$ are parametric eqn's of curve 'C' then

$$\int_C \vec{F} \cdot d\vec{r} = \int_A^B \vec{F} \cdot \frac{d\vec{r}}{dt} dt = \int_A^B \left(\vec{F} \cdot \frac{d\vec{r}}{dt} \right) dt$$

* If 'C' is a closed curve then the line integral is denoted by $\oint \vec{F} \cdot d\vec{r}$

* $\int_C \vec{F} \cdot d\vec{r}$, $\int_C \phi d\vec{r}$ are other types of line integrals.

Application of line integral :-

* The application of line integral is to define the workdone by the force.

If ' $\vec{F}$ ' represents the force vector acting on a particle moving along an arc 'AB'. Then the total workdone by force ' $\vec{F}$ ' during displacement from A to B is given by $\int_A^B \vec{F} \cdot d\vec{r}$

* If the force vector $\vec{F}$ is conservative then there exist a scalar point function ' ϕ ' such that $\vec{F} = \nabla\phi$. Hence we

$$\begin{aligned} \text{workdone} &= \int_C \vec{F} \cdot d\vec{r} \\ &= \int_A^B \nabla\phi \cdot d\vec{r} \\ &= \int_A^B \left(\vec{i} \frac{\partial\phi}{\partial x} + \vec{j} \frac{\partial\phi}{\partial y} + \vec{k} \frac{\partial\phi}{\partial z} \right) \cdot (dx\vec{i} + dy\vec{j} + dz\vec{k}) \\ &= \int_A^B \frac{\partial\phi}{\partial x} dx + \frac{\partial\phi}{\partial y} dy + \frac{\partial\phi}{\partial z} dz \\ &= \int_A^B d\phi = [\phi]_A^B = \phi(B) - \phi(A) \end{aligned}$$

If $\vec{F}$ is conservative then the workdone depends upon the value of ' ϕ ' at the endpoints A and B, not on the path joining A & B.

Note :-

* If $\vec{F}$ is conservative that is $\vec{F} = \nabla\phi$, then $\text{curl } \vec{F} = 0$.

$\Rightarrow \vec{F}$ is irrotational.

Circulation :-

* If $\vec{F}$ represents the velocity of the fluid particle then

$\oint \vec{F} \cdot d\vec{r}$ is called circulation of $\vec{F}$. Where 'C' is a closed curve.

* If $\int_C \vec{F} \cdot d\vec{r} = 0$ then $\vec{F}$ is called irrotational.

1. If $\vec{F} = (5xy - 6x^2)\vec{i} + (2y - ux)\vec{j}$ Evaluate $\int_C \vec{F} \cdot d\vec{r}$ along the curve 'C' in the xy-plane $y = x^3$ from (1,1) to (2,8)

Sol:- Given $\vec{F} = (5xy - 6x^2)\vec{i} + (2y - ux)\vec{j}$

let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, then $d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$
on the xy-plane $z=0$, $dz=0$.

then $d\vec{r} = dx\vec{i} + dy\vec{j}$

$$\begin{aligned}\text{Now } \vec{F} \cdot d\vec{r} &= [(5xy - 6x^2)\vec{i} + (2y - ux)\vec{j}] \cdot (dx\vec{i} + dy\vec{j}) \\ &= (5xy - 6x^2)dx + (2y - ux)dy\end{aligned}$$

Along the curve $y = x^3$

$$\Rightarrow dy = 3x^2 dx$$

and x varies from 1 to 2.

$$\begin{aligned}\therefore \vec{F} \cdot d\vec{r} &= (5xx^3 - 6x^2)dx + (2x^3 - ux)3x^2 dx \\ &= (5x^4 - 6x^2)dx + (6x^5 - 12x^3)dx \\ &= (5x^4 - 6x^2 + 6x^5 - 12x^3)dx\end{aligned}$$

$$\begin{aligned}\therefore \int_C \vec{F} \cdot d\vec{r} &= \int_1^2 (5x^4 - 6x^2 + 6x^5 - 12x^3)dx \\ &= \left[5 \frac{x^5}{5} - \frac{6x^3}{3} + \frac{6x^6}{6} - \frac{12x^4}{4} \right]_1^2 \\ &= [x^5 - 2x^3 + x^6 - 3x^4]_1^2 \\ &= 32(2)^5 - 2(2)^3 + 2^6 - 3(2)^4 - (1^5 - 2(1)^3 + 1^6 - 3(1)^4) \\ &= 32 - 16 + 64 - 48 - (1 - 2 + 1 - 3) \\ &= (32 + 3) \\ &= 35\end{aligned}$$

2. Evaluate $\int_C y^2 dx - 2x^2 dy$ along the parabola $y = x^2$ from (0,0) to (2,4).

Sol:- Given $\int_C y^2 dx - 2x^2 dy$

$$\vec{F} \cdot d\vec{r} = y^2 dx - 2x^2 dy$$

$y = x^2$
 Along the curve $y = x^2$
 $dy = 2x dx$

x varies from 0 to 2

$$\therefore \bar{F} \cdot d\bar{r} = (x^2)^2 dx - 2x^2 \cdot 2x dx$$

$$= x^4 dx - 4x^3 dx$$

$$\int_C \bar{F} \cdot d\bar{r} = \int_0^2 x^4 dx - \int_0^2 4x^3 dx$$

$$= \left[\frac{x^5}{5} \right]_0^2 - \left[\frac{4x^4}{4} \right]_0^2$$

$$= \frac{32}{5} - 2^4$$

$$= \frac{32}{5} - 16$$

$$= -\frac{48}{5} //$$

3. If $\bar{F} = 3xy\bar{i} - y^2\bar{j}$, evaluate $\int_C \bar{F} \cdot d\bar{r}$ where C is curve $y = 2x^2$ in the xy -plane from $(0,0)$ to $(1,2)$

Sol:- Given $\bar{F} = 3xy\bar{i} - y^2\bar{j}$

$$\bar{r} = x\bar{i} + y\bar{j} \quad \text{then } d\bar{r} = dx\bar{i} + dy\bar{j}$$

$$\therefore \bar{F} \cdot d\bar{r} = (3xy\bar{i} - y^2\bar{j}) \cdot (dx\bar{i} + dy\bar{j})$$

$$\bar{F} \cdot d\bar{r} = 3xy dx - y^2 dy$$

Along the curve $y = 2x^2$

$$dy = 4x dx$$

x varies from 0 to 1

$$\bar{F} \cdot d\bar{r} = 3x(2x^2) dx - (2x^2)^2 \cdot 4x dx$$

$$= 6x^3 dx - 16x^5 dx$$

$$\int_C \bar{F} \cdot d\bar{r} = \int_0^1 6x^3 dx - 16 \int_0^1 x^5 dx$$

$$= \left[\frac{6x^4}{4} \right]_0^1 - 16 \left[\frac{x^6}{6} \right]_0^1$$

$$= \frac{3}{2} [x^4]_0^1 - \frac{8}{3} [x^6]_0^1$$

$$= \frac{3}{2} (1)^4 - \frac{8}{3} (1)^6$$

$$= \frac{3}{2} - \frac{8}{3} \Rightarrow \frac{9-16}{6} \Rightarrow -\frac{7}{6} //$$

4. If $\vec{F} = xy\vec{i} - z\vec{j} + x^2\vec{k}$ and C is the curve $x=t^2, y=2t, z=t^3$ from $t=0$ to $t=1$ Evaluate $\int_C \vec{F} \cdot d\vec{r}$.

Sol:- Given $\vec{F} = xy\vec{i} - z\vec{j} + x^2\vec{k}$

Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ then $d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

Now $\vec{F} \cdot d\vec{r} = xydx - zdy + x^2dz$

Given that $x=t^2, y=2t, z=t^3$

$dx = 2t dt, dy = 2 dt, dz = 3t^2 dt$

Integrating of $\vec{F} \cdot d\vec{r}$:

$$\vec{F} \cdot d\vec{r} = t^2(2t)2t dt - t^3(2 dt) + (t^2)^2 3t^2 dt$$

$$= 4t^5 dt - 2t^3 dt + 3t^6 dt$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 4t^5 dt - \int_0^1 2t^3 dt + \int_0^1 3t^6 dt$$

$$= \left[4 \frac{t^6}{6} \right]_0^1 - \left[2 \frac{t^4}{4} \right]_0^1 + \left[3 \frac{t^7}{7} \right]_0^1$$

$$= \frac{4}{6}(1) - \frac{2}{4}(1) + \frac{3}{7}(1)$$

$$= \frac{4}{6} - \frac{1}{2} + \frac{3}{7}$$

$$= \frac{51}{70} //$$

5. Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = (x-3y)\vec{i} + (y-2x)\vec{j}$ where C is the curve in the xy plane $x=2\cos t, y=3\sin t$ from $t=0$ to $t=2\pi$.

Sol:- Given $\vec{F} = (x-3y)\vec{i} + (y-2x)\vec{j}$

Let $\vec{r} = x\vec{i} + y\vec{j}$ then $d\vec{r} = dx\vec{i} + dy\vec{j}$

Now $\vec{F} \cdot d\vec{r} = (x-3y)dx + (y-2x)dy$

G. T. $x=2\cos t, y=3\sin t$

$dx = -2\sin t dt, dy = 3\cos t dt$

then $\vec{F} \cdot d\vec{r} = (2\cos t - 3(3\sin t))(-2\sin t)dt + (3\sin t - 2(2\cos t))3\cos t dt$

$$= (-4\cos t \sin t + 18\sin^2 t + 9\sin t \cos t - 12\cos^2 t) dt$$

$$= (5\cos t \sin t + 18\sin^2 t - 12\cos^2 t) dt$$

$$\int_C \vec{F} \cdot d\vec{r} = 5 \int_0^{2\pi} \cos t \sin t dt + 18 \int_0^{2\pi} \frac{1 - \cos 2t}{2} dt - 12 \int_0^{2\pi} \frac{1 + \cos 2t}{2} dt$$

$$\sin t = z \Rightarrow \cos t dt = dz$$

$$\begin{aligned}
 \int_C \vec{F} \cdot d\vec{r} &= 5 \int_0^{2\pi} \frac{z^2}{2} dz + \frac{18}{2} \int_0^{2\pi} 1 - \cos 2t \, dt + \frac{12}{2} \int_0^{2\pi} 1 + \cos 2t \, dt \\
 &= 5 \left[\frac{z^3}{6} \right]_0^{2\pi} + 9 \left[t \right]_0^{2\pi} - 9 \left[\frac{\sin 2t}{2} \right]_0^{2\pi} + \frac{12}{2} \left[t \right]_0^{2\pi} + \frac{12}{2} \left[\frac{\sin 2t}{2} \right]_0^{2\pi} \\
 &= \frac{5}{2} (\sin^2 2t)_0^{2\pi} + 9(2\pi - 0) - \frac{9}{2} (\sin 4\pi - 0) + \frac{12}{2} (2\pi) + \frac{12}{4} (\sin 4\pi - 0) \\
 &= \frac{5}{2} (\sin^2 2\pi - 0) + 18\pi - \frac{9}{2} \sin 0 + 12\pi + \frac{12}{4} (0) \\
 &= \frac{5}{2} (0) + 18\pi + 0 - 0 + 12\pi \\
 &= 30\pi // \\
 &= 6\pi
 \end{aligned}$$

6. Find the work done in moving a particle in the force field $\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k}$ along the curve $x = 2t^2$, $y = t$, $z = 4t^2 - t$ from $t = 0$ to $t = 1$.

Soln - Given $\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k}$

let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ then $d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$$\vec{F} \cdot d\vec{r} = 3x^2 dx + (2xz - y) dy + z dz$$

Given that

$$\begin{aligned}
 x &= 2t^2, & y &= t, & z &= 4t^2 - t \\
 dx &= 4t \, dt, & dy &= dt, & dz &= (8t - 1) \, dt
 \end{aligned}$$

integral in terms of t .

$$\begin{aligned}
 \vec{F} \cdot d\vec{r} &= 3(2t^2)^2 4t \, dt + (2(2t^2)(4t^2 - t) - t) dt + (4t^2 - t)(8t - 1) dt \\
 &= 48t^5 dt + (16t^4 + 28t^3 - 12t^2) dt + (32t^3 - 4t^2 - 8t^2 + t) dt
 \end{aligned}$$

of t are 0 to 1

$$\begin{aligned}
 \int_C \vec{F} \cdot d\vec{r} &= \int_0^1 (48t^5 + 16t^4 + 28t^3 - 12t^2) dt \\
 &= \left[48 \frac{t^6}{6} + 16 \frac{t^5}{5} + 28 \frac{t^4}{4} - 12 \frac{t^3}{3} \right]_0^1 \\
 &= \left[8t^6 + \frac{16}{5}t^5 + 7t^4 - 4t^3 \right]_0^1 = \left(8 + \frac{16}{5} + 7 - 4 \right) \\
 &= 11 + \frac{16}{5} = \frac{71}{5} //
 \end{aligned}$$

7. Find the workdone by a force $\vec{F} = 2xy\vec{i} - 4z\vec{j} + 5x\vec{k}$ along the curve $x=t^2$, $y=2t+1$, $z=t^3$ from $t=1$ to $t=2$.

8. Find the workdone by the force $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$ when it moves a particle along the arc of the curve $\vec{r} = \cos t\vec{i} + \sin t\vec{j} - t\vec{k}$ from $t=0$ to $t=2\pi$.

Sol:- Given $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$

$$\text{let } \vec{r} = \cos t\vec{i} + \sin t\vec{j} - t\vec{k}$$

i.e ; $x = \cos t$, $y = \sin t$, $z = -t$ are the parametric eqn's of the curve.

$$\text{let } \vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \text{ then } d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\vec{F} \cdot d\vec{r} = zdx + xdy + ydz$$

$$\begin{aligned} x &= \cos t & y &= \sin t & z &= -t \\ dx &= -\sin t dt & dy &= \cos t dt & dz &= -dt \end{aligned}$$

$$\begin{aligned} \vec{F} \cdot d\vec{r} &= -t(-\sin t)dt + \cos t \cos t dt + \sin t(-dt) \\ &= (t \sin t + \cos^2 t - \sin t) dt \end{aligned}$$

the limits of t are 0 to 2π

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} (t \sin t + \cos^2 t - \sin t) dt \\ &= \int_0^{2\pi} t \sin t dt + \int_0^{2\pi} \frac{1 + \cos 2t}{2} dt - \int_0^{2\pi} \sin t dt \\ &= \left[t(-\cos t) \right]_0^{2\pi} - \int_0^{2\pi} \cos t dt + \frac{1}{2} \left[\int_0^{2\pi} dt + \int_0^{2\pi} \cos 2t dt \right] + \left[\cos t \right]_0^{2\pi} \\ &= -\left(2\pi \cos 2\pi - 0 \right) + \left[\sin t \right]_0^{2\pi} + \frac{1}{2} \left[\left(t \right)_0^{2\pi} + \left[\frac{\sin 2t}{2} \right]_0^{2\pi} \right] + \cos 2\pi - \cos 0 \\ &= -2\pi \cos 2\pi + \sin 2\pi - \sin 0 + \frac{1}{2} 2\pi + \frac{1}{4} (\sin 4\pi - \sin 0) + \cos 2\pi - \cos 0 \end{aligned}$$

$$= -2\pi (1) + 0 - 0 + \pi + \frac{1}{4} (0) - \frac{1}{4} (0) + 1 - 1$$

$$= -2\pi + \pi$$

$$= -\pi //$$

$$\begin{aligned} \because \sin n\pi &= 0 \\ \cos n\pi &= (-1)^n \end{aligned}$$

⑦ Sol:- Given $\vec{F} = 2xy\vec{i} - 4z\vec{j} + 5x\vec{k}$
 let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ then $d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$
 $\vec{F} \cdot d\vec{r} = 2xydx - 4zdy + 5xdz$

G.T. $x = t^2$, $y = 2t+1$, $z = t^3$
 $dx = 2t dt$, $dy = 2 dt$, $dz = 3t^2 dt$

Now $\vec{F} \cdot d\vec{r} = 2t^2(2t+1)(2t dt) - 4t^3(2 dt) + 5t^2(3t^2 dt)$
 $= (4t^3(2t+1) - 8t^3 + 15t^4) dt$
 $= (8t^4 + 4t^3 - 8t^3 + 15t^4) dt$
 $= 15t^4 dt$, the limits of 't' are 1 to 2

$$\int_C \vec{F} \cdot d\vec{r} = 15 \int_1^2 t^4 dt$$

$$= 15 \left[\frac{t^5}{5} \right]_1^2$$

$$= 3(2^5 - 1)$$

$$= 3(32 - 1)$$

$$= 3(31)$$

$$= 93 //$$

⑦ Sol:- Given $\vec{F} = 2xy\vec{i} - 4z\vec{j} + 5x\vec{k}$

let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ then $d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$$\vec{F} \cdot d\vec{r} = 2xydx - 4zdy + 5xdz$$

G.T. $x = t^2$, $y = 2t+1$, $z = t^3$

$$dx = 2t dt$$
, $dy = 2 dt$, $dz = 3t^2 dt$

Now $\vec{F} \cdot d\vec{r} = 2t^2(2t+1)2t dt - 4t^3 2 dt + 5t^2(3t^2) dt$

$$= 4t^3(2t+1) dt - 8t^3 dt + 15t^4 dt$$

$$= (8t^4 + 4t^3 - 8t^3 + 15t^4) dt$$

$$= (23t^4 - 4t^3) dt$$
, the limits of 't' are 1 to 2

$$\int_C \vec{F} \cdot d\vec{r} = \int_1^2 (23t^4 - 4t^3) dt$$

$$\begin{aligned}
 &= \left(23 \frac{t^5}{5} - u \frac{t^4}{4} \right) \Big|_1^{23} \\
 &= \left(\frac{23}{5} 25 - 24 \right) - \left(\frac{23}{5} (1)^5 - (1)^4 \right) \\
 &= \left(\frac{23}{5} (32) - 16 \right) - \left(\frac{23}{5} - 1 \right) \\
 &= \left(\frac{736}{5} - 16 \right) - \left(\frac{23-5}{5} \right) \\
 &= \frac{736-80}{5} - \frac{18}{5} \\
 &= \frac{656}{5} - \frac{18}{5} \Rightarrow \frac{638}{5} //
 \end{aligned}$$

Q8. Find the workdone in moving a field particle in the force field. $\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k}$ along the straight line from $(0,0,0)$ to $(2,1,3)$.

Sol:- Given $\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k}$

let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ then $d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$$\vec{F} \cdot d\vec{r} = 3x^2 dx + (2xz - y) dy + z dz$$

To find $\int_C \vec{F} \cdot d\vec{r}$:- Here 'c' is the straight line joining $O(0,0,0)$ and $A(2,1,3)$.

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

$$\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0}$$

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t \text{ (say)}$$

$$x = 2t, \quad y = t, \quad z = 3t$$

$$dx = 2 dt, \quad dy = dt, \quad dz = 3 dt$$

$$\vec{F} \cdot d\vec{r} = 3(2t)^2 2 dt + (2(2t)(3t) - t) dt + 3t(3 dt)$$

$$= (24t^2 + 12t^2 - t + 9t) dt$$

$$= (36t^2 + 8t) dt$$

$$= (36t^2 + 8t) dt$$

At $(0,0,0)$, $x=0$, $y=0$, $z=0$ then $t=0$

At $(2,1,3)$, $x=2$, $y=1$, $z=3$ then $t=1$

then work done

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_0^1 (36t^2 + 8t) dt \\ &= \left(36 \frac{t^3}{3} + 8 \frac{t^2}{2} \right)_0^1 \\ &= (12t^3 + 4t^2)_0^1 \\ &= 12(1) + 4(1) \\ &= 16\end{aligned}$$

10. If $\vec{F} = (x^2 + y^2)\vec{i} - 2xy\vec{j}$, And $\int_C \vec{F} \cdot d\vec{r}$ where C is the rectangular in xy -plane bounded by the lines $y=0, y=b, x=0, x=a$

SOL:- Given $\vec{F} = (x^2 + y^2)\vec{i} - 2xy\vec{j}$

let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ then $d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$$\vec{F} \cdot d\vec{r} = (x^2 + y^2)dx - 2xydy$$

To find $\int_C \vec{F} \cdot d\vec{r}$:- Here 'C' is a rectangle formed by the lines

$$y=0, y=b, x=0, x=a$$

the curve 'C' consist of four lines OA, AB, BC, CD

$$C = C_1 \cup C_2 \cup C_3 \cup C_4$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{OA} \vec{F} \cdot d\vec{r} + \int_{AB} \vec{F} \cdot d\vec{r} + \int_{BC} \vec{F} \cdot d\vec{r} + \int_{CO} \vec{F} \cdot d\vec{r} \rightarrow \textcircled{1}$$

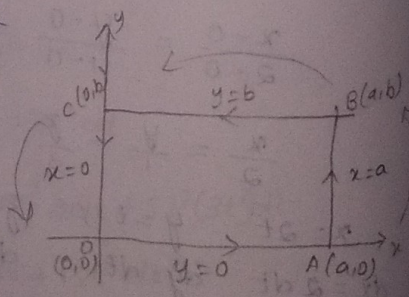
Along the line OA :- The eqn of OA is $y=0, dy=0$

$$\text{then } \vec{F} \cdot d\vec{r} = x^2 dx - 2x(0) = x^2 dx$$

and x varies from 0 to a

$$\int_{OA} \vec{F} \cdot d\vec{r} = \int_{x=0}^a x^2 dx$$

$$= \left(\frac{x^3}{3} \right)_0^a \Rightarrow \frac{a^3}{3}$$



Along the line AB :- The eqn of AB is $x=a, dx=0$

$$\vec{F} \cdot d\vec{r} = x^2 + y^2(0) - 2aydy \Rightarrow -2aydy$$

and y varies from 0 to b

$$\int_{AB} \vec{F} \cdot d\vec{r} = \int_0^b -2aydy \Rightarrow -2a \left(\frac{y^2}{2} \right)_0^b \Rightarrow -ab^2$$

Along the line BC:- The eqn of BC is $y=b$, $dy=0$

$$\vec{F} \cdot d\vec{r} = (x^2 + b^2) dx - 2x(0) d0$$

$$= (x^2 + b^2) dx$$

x varies from a to 0

$$\int_{BC} \vec{F} \cdot d\vec{r} = \int_a^0 (x^2 + b^2) dx$$

$$= \left(\frac{x^3}{3} + b^2 x \right)_a^0$$

$$= 0 - \frac{a^3}{3} + b^2(0 - a)$$

$$= -\frac{a^3}{3} - ab^2 \Rightarrow -\left(\frac{a^3}{3} + ab^2\right)$$

$\therefore$ Along the line CO:- The eqn of CO is $x=0$, $dx=0$

$$\vec{F} \cdot d\vec{r} = (0 + y^2)(0) - 2(0)y dy = 0$$

y varies from b to 0

$$\int_{CO} \vec{F} \cdot d\vec{r} = 0$$

from (1)

$$\int_C \vec{F} \cdot d\vec{r} = \frac{a^3}{3} - ab^2 - \left(\frac{a^3}{3} + ab^2\right)$$

$$= -ab^2 - ab^2$$

$$= -2ab^2 \text{ h.}$$

11. Compute the line integral $\oint_C y^2 dx - x^2 dy$ around the triangle whose vertices are $(1,0)$, $(0,1)$, $(-1,0)$ in xy -plane.

SOL:-

12. Evaluate the line integral $\int_C (x^2 + xy) dx + (x^2 + y^2) dy$ where C is the square formed by the lines $x=\pm 1$, $y=\pm 1$.

13. Find the circulation $\vec{F} = (2x - y + 2z)\vec{i} + (x + y - z)\vec{j} + (3x - 2y - 5z)\vec{k}$ along the circle $x^2 + y^2 = 4$ in the xy -plane.

SOL:- Given $\vec{F} = (2x - y + 2z)\vec{i} + (x + y - z)\vec{j} + (3x - 2y - 5z)\vec{k}$

Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ then $d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

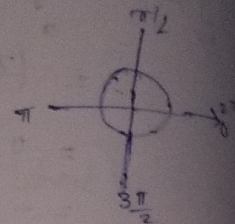
$\vec{F} \cdot d\vec{r} = 2x dx - y dy + 2z dz + x dx + y dy - z dz + 3x dx - 2y dy - 5z dz$
on the xy -plane $z=0$ so that
 $d\vec{r} = dx\vec{i} + dy\vec{j}$

$$\text{Now } \vec{F} \cdot d\vec{r} = (2x - y + 0)dx + (x + y - 0)dy + 0 \\ = (2x - y)dx + (x + y)dy$$

$\therefore$ Circulation = $\oint_C \vec{F} \cdot d\vec{r}$, where 'C' is the circle $x^2 + y^2 = 4$ with centre (0,0) and radius 2.

The parametric eqn of circle

$$\begin{aligned} x &= r \cos \theta, & y &= r \sin \theta \\ x &= 2 \cos \theta, & y &= 2 \sin \theta \\ dx &= -2 \sin \theta d\theta, & dy &= 2 \cos \theta d\theta. \end{aligned}$$



θ varies from 0 to 2π

$$\begin{aligned} \therefore \oint_C \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} (2(2 \cos \theta) - 2 \sin \theta)(-2 \sin \theta) d\theta + (2 \cos \theta + 2 \sin \theta)(2 \cos \theta) d\theta \\ &= (4 \cos \theta - 2 \sin \theta)(-2 \sin \theta) d\theta + (2 \cos \theta + 2 \sin \theta)(2 \cos \theta) d\theta \\ &= (-8 \cos \theta \sin \theta + 4 \sin^2 \theta + 4 \cos^2 \theta + 4 \cos \theta \sin \theta) d\theta \\ &= (4(\sin^2 \theta + \cos^2 \theta) - 4 \cos \theta \sin \theta) d\theta \\ &= (4 - 4 \sin \theta \cos \theta) d\theta \end{aligned}$$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (4 - 2 \sin 2\theta) d\theta$$

$$= \int_0^{2\pi} (4 - 2 \sin 2\theta) d\theta$$

$$= 4[0]_0^{2\pi} - 2 \left[\frac{-\cos 2\theta}{2} \right]_0^{2\pi}$$

$$= 4(2\pi) + (\cos 4\pi - \cos 0)$$

$$= 8\pi + (1 - 1)$$

$$= 8\pi //$$

14. P.T the force field $\vec{F} = 2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^2\vec{k}$ is conservative and find the scalar potential. Find the work done in moving a particle from (1, -1, 2) to (3, 2, -1)

Sol:- Given $\vec{F} = 2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^2\vec{k}$

Let a vector field $\vec{F}$ is called conservative if $\nabla \times \vec{F} = 0$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz^3 & x^2z^3 & 3x^2yz^2 \end{vmatrix}$$

$$= \vec{i} \left(\frac{\partial}{\partial y} 3x^2yz^2 - \frac{\partial}{\partial z} x^2z^3 \right) - \vec{j} \left(\frac{\partial}{\partial x} 3x^2yz^2 - \frac{\partial}{\partial z} 2xyz^3 \right) + \vec{k} \left(\frac{\partial}{\partial x} x^2z^3 - \frac{\partial}{\partial y} 2xyz^3 \right)$$

$$= \vec{i} (3x^2z^2 - 3x^2z^2) - \vec{j} (6xyz^2 - 6xyz^2) + \vec{k} (2xz^3 - 2xz^3) \\ = \vec{i}(0) - \vec{j}(0) + \vec{k}(0)$$

$$\nabla \times \vec{F} = 0$$

$\therefore \vec{F}$ is conservative.

Let ' ϕ ' is the scalar potential of $\vec{F}$ then

$$\vec{F} = \nabla \phi$$

$$2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^2\vec{k} = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}$$

comparing the coefficients of $\vec{i}, \vec{j}, \vec{k}$

$$\frac{\partial \phi}{\partial x} = 2xyz^3 \rightarrow (1), \quad \frac{\partial \phi}{\partial y} = x^2z^3 \rightarrow (2), \quad \frac{\partial \phi}{\partial z} = 3x^2yz^2$$

Integrating eqn (1) & (2) & (3) partially x, y, z we get

$$(1) \Rightarrow \phi = \int 2xyz^3 dx \\ = 2yz^3 \frac{x^2}{2} + c_1(x, y, z) \\ = x^2yz^3 + c_1(x, y, z)$$

$$(2) \Rightarrow \phi = x^2yz^3 + c_2(x, z)$$

$$(3) \Rightarrow \phi = x^2yz^3 + c_3(x, y)$$

$\therefore$ hence the repeated terms

$$\phi = x^2yz^3 + k$$

Since $\vec{F}$ is conservative then the workdone depends upon the value ' ϕ ' at the end points.

$$(1, -1, 2) \text{ and } (3, 2, -1)$$

$$\therefore \text{Workdone} = \int_A^B \vec{F} \cdot d\vec{r}$$

$$\begin{aligned}
 &= \int_A^B d\phi \\
 &= [\phi]_A^B \\
 &= [x^2 y z^3]_{(1, -1, 2)}^{(3, 2, -1)} \\
 &= (9 \cdot 2 \cdot (-1)) - (1 \cdot (-1) \cdot (2)^3) \\
 &= -18 + 8 \\
 &= -10
 \end{aligned}$$

15. If $\vec{F} = 2y\vec{i} - z\vec{j} + x\vec{k}$. Find $\int_C \vec{F} \cdot d\vec{r}$ where 'C' is the curve $x = \cos t$, $y = \sin t$ and $z = 2\cos t$ from $t=0$ to $t=\frac{\pi}{2}$.

Sol:- Given the $\vec{F} = 2y\vec{i} - z\vec{j} + x\vec{k}$

Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ then $d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$$\vec{F} \cdot d\vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2y & -z & x \\ dx & dy & dz \end{vmatrix}$$

$$= \vec{i} [-z dz - x dy] - \vec{j} [2y dz - x dx] + \vec{k} [2y dy + z dx]$$

$x = \cos t$

$x = \cos t$

$y = \sin t$

$z = 2\cos t$

$dx = -\sin t dt$

$dy = \cos t dt$

$dz = -2\sin t dt$

$$\vec{F} \cdot d\vec{r} = \vec{i} [-2\cos t (-2\sin t dt) - \cos t \cos t dt] - \vec{j} [2\sin t (-2\sin t dt) - \cos t (-\sin t) dt] + \vec{k} [2\sin t \cos t dt + 2\cos t (-\sin t) dt]$$

$$= \vec{i} [4\cos t \sin t dt - \cos^2 t dt] - \vec{j} [-4\sin^2 t dt + \cos t \sin t dt] + \vec{k} [2\sin t \cos t dt - 2\cos t \sin t dt]$$

$$= \vec{i} (4\sin t \cos t - \cos^2 t) dt - \vec{j} (-4\sin^2 t + \sin t \cos t) dt + \vec{k} (2\sin t \cos t - 2\cos t \sin t) dt$$

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi/2} [\vec{i} (4\sin t \cos t - \cos^2 t) dt - \vec{j} (-4\sin^2 t + \sin t \cos t) dt + \vec{k} (2\sin t \cos t - 2\cos t \sin t) dt]$$

$$= \vec{i} \left(2 \int_0^{\pi/2} \sin 2t dt - \int_0^{\pi/2} \frac{1 + \cos 2t}{2} dt \right) - \vec{j} \left(-4 \int_0^{\pi/2} \sin^2 t dt + \int_0^{\pi/2} \frac{\sin 2t}{2} dt \right) + \vec{k} \left(\int_0^{\pi/2} \sin 2t dt - \int_0^{\pi/2} \sin 2t dt \right)$$

$$\begin{aligned}
 \int_C \vec{F} \cdot d\vec{r} &= \vec{i} \left[2 \left(\frac{-\cos 2t}{2} \right) \Big|_0^{\pi/2} - \frac{1}{2} \left(t \right) \Big|_0^{\pi/2} + \left(\frac{\sin 2t}{2} \right) \Big|_0^{\pi/2} \right] - \vec{j} \left[-\frac{1}{2} \left(t \right) \Big|_0^{\pi/2} - \left(\frac{\sin 2t}{2} \right) \Big|_0^{\pi/2} \right] \\
 &\quad + \frac{1}{2} \left(\frac{-\cos 2t}{2} \right) \Big|_0^{\pi/2} \\
 &= \vec{i} \left[(-\cos \pi + \cos 0) - \frac{1}{2} \left(\frac{\pi}{2} - 0 \right) + \frac{1}{2} (\sin \pi - \sin 0) \right] - \vec{j} \left[-\frac{1}{2} \left(\frac{\pi}{2} - 0 \right) - \frac{1}{2} (\sin \pi - \sin 0) \right] \\
 &\quad + \frac{1}{4} (-\cos \pi + \cos 0) \\
 &= \vec{i} \left[(1+1) - \frac{\pi}{4} + \frac{1}{4} (0-0) \right] - \vec{j} \left[-\frac{\pi}{4} + \frac{1}{4} (0-0) - \frac{1}{4} (-1-1) \right] \\
 &= \vec{i} \left(2 - \frac{\pi}{4} \right) - \vec{j} \left(-\pi + \frac{1}{2} \right) \\
 &= \left(2 - \frac{\pi}{4} \right) \vec{i} + \left(\pi - \frac{1}{2} \right) \vec{j}
 \end{aligned}$$

16. If $\phi = x^2 y z^3$, evaluate $\int_C \phi \, d\vec{r}$ along the curve $x=t, y=t^2, z=t^3$ from $t=0$ to $t=1$.

Sol:- Given $\phi = x^2 y z^3$

Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ then $d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$$\begin{aligned}
 \text{Now } \phi d\vec{r} &= x^2 y z^3 (dx\vec{i} + dy\vec{j} + dz\vec{k}) \\
 &= x^2 y z^3 dx\vec{i} + x^2 y z^3 dy\vec{j} + x^2 y z^3 dz\vec{k}
 \end{aligned}$$

$$\begin{aligned}
 \text{Given } x &= t, \quad y = t^2, \quad z = t^3 \\
 dx &= dt, \quad dy = 2t \, dt, \quad dz = 3t^2 \, dt
 \end{aligned}$$

$$\begin{aligned}
 \phi d\vec{r} &= t^2 t^2 (t^3)^3 dt \vec{i} + t^2 t^2 (t^3)^3 2t \, dt \vec{j} + t^2 t^2 (t^3)^3 3t^2 \, dt \vec{k} \\
 &= t^{13} dt \vec{i} + 2t^{14} dt \vec{j} + 3t^{15} dt \vec{k}
 \end{aligned}$$

$$\begin{aligned}
 \int_C \phi d\vec{r} &= \int_0^1 t^{13} dt \vec{i} + 2 \int_0^1 t^{14} dt \vec{j} + 3 \int_0^1 t^{15} dt \vec{k} \\
 &= \vec{i} \left[\frac{t^{14}}{14} \right]_0^1 + 2 \vec{j} \left[\frac{t^{15}}{15} \right]_0^1 + 3 \vec{k} \left[\frac{t^{16}}{16} \right]_0^1
 \end{aligned}$$

$$= \frac{1}{14} (1^{14} - 0) + 2 \vec{j} \frac{1}{15} (1^{15} - 0) + \frac{3}{16} \vec{k} (1^{16} - 0)$$

$$\int_C \phi d\vec{r} = \frac{1}{14} \vec{i} + \frac{2}{15} \vec{j} + \frac{3}{16} \vec{k}$$

Green's theorem:-

Let 'R' is closed region in xy-plane bounded by a simple closed curve 'C' and if m & n are functions of x & y having partial derivatives. $\frac{\partial M}{\partial y}$ & $\frac{\partial N}{\partial x}$,

$$\frac{\partial M}{\partial y} \text{ \& } \int_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

where 'C' is traversed in anticlockwise direction.

Note:-

* ^{With} Green's theorem establishes relation b/w line & double integrals.

* If $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$ then $\int_C M dx + N dy = 0$

* If $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = k$, where 'k' is a constant then

$$\begin{aligned} \int_C M dx + N dy &= k \iint_R dx dy \\ &= k (\text{Area of region R}) \end{aligned}$$

1. Using Green's theorem, evaluate $\int_C (2xy - x^2) dx + (x^2 + y^2) dy$ where 'C' is the closed curve of the region bounded by $y = x^2$ & $y = x$

Sol:- Given 'C' is a closed curve formed by

$$y = x^2 \text{ \& } y = x \rightarrow \textcircled{C}$$

$$y^2 = x^4$$

$$x = x^4$$

$$x - x^4 = 0$$

$$x(1 - x^3) = 0$$

$$x = 0, 1 - x^3 = 0$$

$$x^3 = 1$$

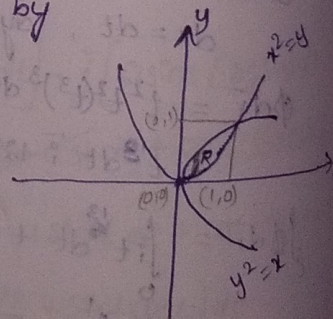
$$x = 1$$

$$\text{at } x = 0 \Rightarrow y = 0$$

$$\text{at } x = 1 \Rightarrow y = 1$$

The ^{curves} points of intersection at (0,0) & (1,1)

$\therefore$ The curves intersect at (0,0) & (1,1)



by Green's theorem

$$\int_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

The given integral is

$$\int_C (2xy - x^2) dx + (x^2 + y^2) dy =$$

here $M = 2xy - x^2$, $\frac{\partial N}{\partial x} = x^2 + y^2$

$$\frac{\partial M}{\partial y} = 2x$$

$$\frac{\partial N}{\partial x} = 2xy$$

x varies from 0 to 1

y varies from x^2 to $\sqrt{x}$

$$\int_C (2xy - x^2) dx + (x^2 + y^2) dy = \int_{x=0}^1 \int_{y=x^2}^{\sqrt{x}} (2x - 2x) dx dy$$

$$= 0$$

Q. Apply Green's theorem through to evaluate $\oint_C (2x^2 - y^2) dx + (x^2 + y^2) dy$ where 'C' is the boundary of the area enclosed by the x-axis and upper half of the circle $x^2 + y^2 = a^2$.

Sol:- Given 'C' is a closed curve formed by the x-axis and upper half of the circle $x^2 + y^2 = a^2$

Given integral is

$$\oint_C (2x^2 - y^2) dx + (x^2 + y^2) dy$$

here $M = 2x^2 - y^2$

$$N = x^2 + y^2$$

$$\frac{\partial M}{\partial y} = -2y$$

$$\frac{\partial N}{\partial x} = 2x$$

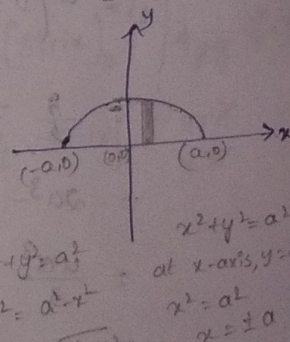
x varies from $-a$ to a

y varies from 0 to $\sqrt{a^2 - x^2}$

∴ by Green's theorem $\int_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$

$$\int_C (2x^2 - y^2) dx + (x^2 + y^2) dy = \int_{x=-a}^a \int_{y=0}^{\sqrt{a^2-x^2}} (2x + 2y) dx dy$$

$$= 2 \int_{-a}^a \left(\int_0^{\sqrt{a^2-x^2}} (2x + 2y) dy \right) dx$$



$$\begin{aligned}
&= 2 \int_{-a}^a \left[xy + \frac{y^2}{2} \right]_0^{\sqrt{a^2-x^2}} dx \\
&= 2 \int_{-a}^a \left(x\sqrt{a^2-x^2} + \frac{(\sqrt{a^2-x^2})^2}{2} - 0 - 0 \right) dx \\
&= 2 \int_{-a}^a x\sqrt{a^2-x^2} + \frac{(a^2-x^2)}{2} dx \\
&= 2 \left[\int_{-a}^a x\sqrt{a^2-x^2} + \frac{1}{2} \int_{-a}^a (a^2-x^2) dx \right] \\
&= 2 \left[\int_{-a}^a x\sqrt{a^2-x^2} + \frac{1}{2} \int_{-a}^a (a^2-x^2) dx \right] \\
&= -\int_{-a}^a (a^2-x^2) \frac{1}{2} dx + \int_{-a}^a (a^2-x^2) dx \\
&= -\left[\frac{(a^2-x^2)^{3/2}}{3/2} \right]_{-a}^a + \left[a^2x - \frac{x^3}{3} \right]_{-a}^a \\
&= -\left[\frac{(a^2-a^2)^{3/2}}{3/2} - \frac{(a^2+a^2)^{3/2}}{3/2} \right] + \left(a^2(a) - \frac{a^3}{3} \right) - \left(-a^2(a) + \frac{a^3}{3} \right) \\
&= -\left[0 - \frac{(2a^2)^{3/2}}{3/2} \right] + \left(a^3 - \frac{a^3}{3} + a^3 - \frac{a^3}{3} \right) \\
&= \frac{8}{3} - \frac{2}{3}(0) + 2a^3 - \frac{2a^3}{3} \\
&= 2a^3 - \frac{2a^3}{3} \\
&= \frac{4a^3}{3}
\end{aligned}$$

3. Evaluate by Green's theorem $\oint_C (y - \sin x) dx + \cos x dy$ where 'C' is the triangle enclosed by the lines $y=0$, $x=\frac{\pi}{2}$, $\pi y=2x$.

Sol:- Given 'C' is the closed curve formed by the lines $y=0$, $x=\frac{\pi}{2}$, $y=\frac{2}{\pi}x$.

Given integral is

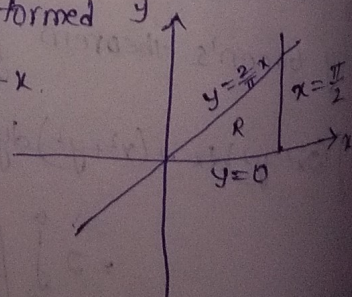
$$\oint_C (y - \sin x) dx + \cos x dy$$

$$M = y - \sin x$$

$$N = \cos x$$

$$\frac{\partial M}{\partial y} = 1$$

$$\frac{\partial N}{\partial x} = -\sin x$$



x varies from 0 to $\frac{\pi}{2}$
y varies from 0 to $\frac{2}{\pi}x$

by Green's theorem

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

$$\oint_C (y - \sin x) dx + \cos x dy = \int_0^{\pi/2} \int_0^{2\pi x} (-\sin x - 1) dx dy$$

$$= \int_0^{\pi/2} \int_0^{2\pi x} (\sin x + 1) dy dx$$

$$= \int_0^{\pi/2} \left[\sin x \left(y \right)_0^{2\pi x} + \left(y \right)_0^{2\pi x} \right] dx$$

$$= \int_0^{\pi/2} \left(\sin x \left(\frac{2x}{\pi} \right) + \frac{2x}{\pi} \right) dx$$

$$= \int_0^{\pi/2} \left(\frac{2}{\pi} \sin x \cdot x + \frac{2}{\pi} x \right) dx$$

$$= -\frac{2}{\pi} \int_0^{\pi/2} (\sin x \cdot x + x) dx$$

$$= -\frac{2}{\pi} \left[x \int_0^{\pi/2} \sin x dx - \int_0^{\pi/2} 1 \cdot \int_0^{\pi/2} \sin x dx + \left(\frac{x^2}{2} \right)_0^{\pi/2} \right]$$

$$= -\frac{2}{\pi} \left[(-x \cos x)_0^{\pi/2} + \int_0^{\pi/2} \cos x dx \right] + \frac{1}{2} \left(\frac{\pi}{2} \right)^2$$

$$= -\frac{2}{\pi} \left[\left(-\frac{\pi}{2} \cos \frac{\pi}{2} + 0 \right) + (\sin x)_0^{\pi/2} \right] + \frac{1}{2} \frac{\pi^2}{4}$$

$$= -\frac{2}{\pi} \left[\left(-\frac{\pi}{2} (0) \right) + \sin \frac{\pi}{2} - \sin 0 \right] - \frac{\pi^2}{4}$$

$$= -\frac{2}{\pi} [0 + 1] - \frac{\pi}{4}$$

$$= -\frac{2}{\pi} - \frac{\pi}{4}$$

4. Verify Green's theorem for $\int_C (xy + y^2) dx + x^2 dy$ where 'C' is bounded by $y=x$ & $y=x^2$.

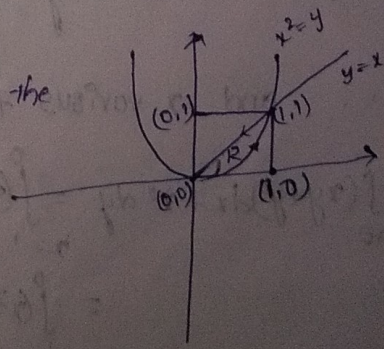
Sol:- Given 'C' is part formed by the curves $y=x \rightarrow (1)$ & $y=x^2$ from (1) & (2)

$$x = x^2$$

$$x - x^2 = 0$$

$$x(1-x) = 0$$

$$x=0, x=1$$



at $x=0 \Rightarrow y=0$, $(x,y) = (0,0)$

at $x=1 \Rightarrow y=1$, $(x,y) = (1,1)$

The curves $y=x$ and $y=x^2$ intersects at $(0,0)$ and $(1,1)$

$\therefore$ by Green's theorem

$$\int_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

$$\text{To find } \int_C M dx + N dy = \int_C (xy + y^2) dx + x^2 dy$$

Here 'C' consists of two curves OA & AD

along OA :- equation of curve OA is $y=x^2$
x varies from 0 to 1

$$y = x^2$$

$$dy = 2x dx$$

$$\int_{OA} (xy + y^2) dx + x^2 dy = \int_{x=0}^1 (x(x^2) + (x^2)^2) dx + x^2 2x dx$$

$$= \int_0^1 (x^3 + x^4) dx + 2x^3 dx$$

$$= \int_0^1 x^3 dx + \int_0^1 x^4 dx + \int_0^1 2x^3 dx$$

$$= \left[\frac{x^4}{4} \right]_0^1 + \left[\frac{x^5}{5} \right]_0^1 + 2 \left[\frac{x^4}{4} \right]_0^1$$

$$= \frac{1}{4} (1)^4 + \frac{1}{5} + \frac{2}{4} (1)$$

$$= \frac{3}{4} + \frac{1}{5} = \frac{19}{20}$$

Along AD :- equation of curve is $y=x$

$$dy = dx$$

and x varies from 1 to 0

$$\int_{AD} (xy + y^2) dx + x^2 dy = \int_{x=1}^0 (x(x) + x^2) dx + x^2 dx$$

$$= \int_1^0 (x^2 + x^2) dx + x^2 dx$$

$$= \int_1^0 2x^2 dx + x^2 dx$$

$$= \int_1^0 3x^2 dx$$

$$= 3 \left[\frac{x^3}{3} \right]_1^0$$

$$= -1$$

$$\therefore \int_C M dx + N dy = \int_C (xy + y^2) dx + x^2 dy = \frac{19}{20} - 1$$

$$= -\frac{1}{20} \rightarrow (3)$$

To find $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$

here $M = xy + y^2$, $N = x^2$

$$\frac{\partial M}{\partial y} = x + 2y \quad \frac{\partial N}{\partial y} = 2x$$

x varies from 0 to 1

y varies from x^2 to x

$$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy = \int_0^1 \int_{x^2}^x (2x - (x + 2y)) dx dy$$

$$= \int_0^1 \left(\int_{x^2}^x (2x - x - 2y) dy \right) dx$$

$$= \int_0^1 \left(2xy - xy - 2 \frac{y^2}{2} \right)_{x^2}^x dx$$

$$= \int_0^1 [2x \cdot x^2 - x \cdot x^2 - (x^2)^2 - (2x(x^2) - x(x^2) - (x^4))] dx$$

$$= \int_0^1 (2x^3 - x^3 - x^4 - 2x^3 + x^3 + x^4) dx$$

$$= \int_0^1 (-x^3 + x^4) dx$$

$$= \left[-\frac{x^4}{4} + \frac{x^5}{5} \right]_0^1$$

$$= -\frac{1}{4} + \frac{1}{5} + \frac{0}{4} + \frac{0}{5}$$

$$= \frac{-5+4}{20}$$

$$= -\frac{1}{20} \rightarrow (4)$$

from (3) & (4) Green's theorem is verified //

5. Verify Green's theorem for $\int (3x^2 - 8y^2) dx + (4y - 6xy) dy$ where 'c' is the region bounded by $x=0$, $y=0$, $x+y=1$

Sol:- Given 'c' is region bounded by curves

$$x=0, y=0, x+y=1 \rightarrow (3)$$

from (1) & (2) and (3)

by green's theorem

$$\int_C M dx + \int N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dy dx$$

To find : $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dy dx$

$$M = 3x^2 - 8y^2, \quad N = 4y - 6xy$$

$$\frac{\partial M}{\partial y} = -16y$$

$$\frac{\partial N}{\partial x} = -6y$$

$$\int_0^1 \int_0^{1-x} (-6y + 16y) dx dy = \int_0^1 \left[-\frac{3y}{x} \cdot \frac{y^2}{2} + 16 \cdot \frac{y^2}{2} \right]_0^{1-x} dx$$

$$= \int_0^1 \left[5y^2 \right]_0^{1-x} dx$$

$$= 5 \int_0^1 (1-x)^2 dx$$

$$= 5 \int_0^1 (1+x^2-2x) dx$$

$$= 5 \left[x + \frac{x^3}{3} - 2 \cdot \frac{x^2}{2} \right]_0^1$$

$$= 5 \left(1 + \frac{1}{3} - 1 - 0 + 0 + 0 \right)$$

$$= 5 \left(\frac{1}{3} \right)$$

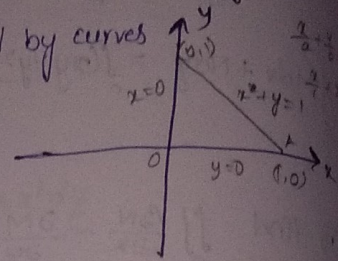
$$= \frac{5}{3}$$

To find $\int_C M dx + N dy = \int_{OA} + \int_{OB} + \int_{BO}$

Along OA, $y=0$

$$dy=0$$

$$x \rightarrow 0 \text{ to } 1$$



$$x=0, 1$$

$$y=1-x$$

$$y=0 \text{ to } 1-x$$

$$\int_0^1 3x^2 dx = 3 \left[\frac{x^3}{3} \right]_0^1$$

$$= 1$$

Along OB :- $y = 1 - x$
 $dy = -dx$

$$\int_1^0 (3x^2 - 8(1-x)^2) dx = (4(1-x) - 6x(1-x)) dx$$

$$= \int_0^1 (3x^2 - 8 + 8x^2 + 16x - 4 + 4x + 6x - 6x^2) dx$$

$$= \int_0^1 (-11x^2 + 26x - 12) dx \Rightarrow \left[-11 \frac{x^3}{3} + 26 \frac{x^2}{2} - 12x \right]_0^1$$

$$= -\frac{11}{3}(1) + 13(1) - 12(1) - \left[-11 \frac{0}{3} + 13(0) - 12(0) \right]$$

$$= -\frac{11}{3} - 13 + 12$$

$$= -\frac{11}{3} - 1$$

$$= -\frac{11+3}{3}$$

$$= -8/3$$

Along BO :- $x = 0$
 $dx = 0$

$$\int_0^1 uy dy = u \left[\frac{y^2}{2} \right]_0^1$$

$$= -2$$

$$\int_C (3x^2 - 8y^2) dx + (4y - 6xy) dy = 1 + \frac{8}{3} - 2$$

$$= \frac{8}{3} - 1 = \frac{5}{3}$$

$$\int_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dy dx$$

$\therefore$ green's theorem is verified.

6. verify green's theorem for $\vec{F} = (x^2 + y^2) \vec{i} - 2xy \vec{j}$ taken around the rectangle bounded by the lines $x=0$, $x=a$, $y=0$, $y=b$.

Sol:- Given $\vec{F} = (x^2 + y^2) \vec{i} - 2xy \vec{j}$

let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ then $d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

in x - y -plane $z=0$ then $dz=0$

$$\therefore d\vec{r} = dx\vec{i} + dy\vec{j}$$

$$\therefore \vec{F} \cdot d\vec{r} = (x^2 + y^2)dx - 2xy dy$$

$$\int_C M dx + N dy = \int_C (x^2 + y^2)dx - 2xy dy$$

by green's theorem

$$\int_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

$$\text{To find } \iint_C M dx + N dy = \int_C (x^2 + y^2)dx - 2xy dy$$

Here 'C' is a rectangle formed by the lines
 $y=0$, $y=b$, $x=0$, $x=a$.

The curve 'C' consist of four lines OA, AB, BC, CD

$$C = C_1 \cup C_2 \cup C_3 \cup C_4$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{OA} \vec{F} \cdot d\vec{r} + \int_{AB} \vec{F} \cdot d\vec{r} + \int_{BC} \vec{F} \cdot d\vec{r} + \int_{CO} \vec{F} \cdot d\vec{r} \rightarrow (1)$$

Along OA :- The eqn of OA is

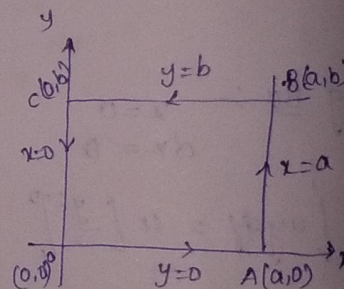
$$y=0$$

$$dy=0$$

$$\vec{F} \cdot d\vec{r} = x^2 dx - 0$$

and x varies from 0 to a

$$\begin{aligned} \int_{OA} \vec{F} \cdot d\vec{r} &= \int_0^a x^2 dx \\ &= \left[\frac{x^3}{3} \right]_0^a = \frac{a^3}{3} \end{aligned}$$



Along AB :- The eqn of AB is $x=a$,

$$dx=0$$

and y varies from 0 to b

$$\begin{aligned} \int_{AB} \vec{F} \cdot d\vec{r} &= \int_0^b -2ay dy \\ &= -2a \left[\frac{y^2}{2} \right]_0^b \\ &= -ab^2 \end{aligned}$$

Along BC :- $y=b$
 $dy=0$

x varies from a to 0

$$\begin{aligned}\int_{BC} \vec{F} \cdot d\vec{r} &= \int_a^0 (x^2 + b^2) dx \\ &= \left[\frac{x^3}{3} + b^2 x \right]_a^0 \\ &= 0 + 0 - \frac{a^3}{3} - b^2 a \\ &= -\left(\frac{a^3}{3} + ab^2 \right)\end{aligned}$$

Along CO :- $x=0$
 $dx=0$
 y varies from b to 0

$$\int_{CO} \vec{F} \cdot d\vec{r} = 0$$

from eqn (1)

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \frac{a^3}{3} - ab^2 - \frac{a^3}{3} + ab^2 \\ &= -2ab^2 \quad \parallel \rightarrow (1)\end{aligned}$$

To find $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$

where 'R' is the region bounded by the lines $x=0$, $x=a$
 $y=0$, $y=b$

$$\begin{aligned}M &= x^2 + y^2 & N &= -2xy \\ \frac{\partial M}{\partial y} &= 2y & \frac{\partial N}{\partial x} &= -2y\end{aligned}$$

$$\begin{aligned}\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy &= \iint_R -2y - 2y \, dx dy \\ &= \iint_R -4y \, dx dy\end{aligned}$$

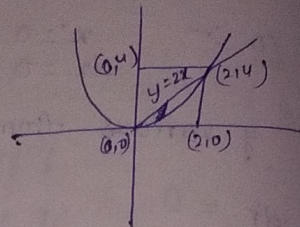
Limits :- x varies from 0 to a (in x -axis)
 y varies from 0 to b

$$\begin{aligned}\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy &= \int_{x=0}^a \int_{y=0}^b -4y \, dy dx \\ &= -4 \int_0^a \left[\frac{y^2}{2} \right]_0^b dx \\ &= -2 \int_0^a b^2 dx \\ &= -2b^2 \left[x \right]_0^a \\ &= -2ab^2\end{aligned}$$

$$\oint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy = -2ab^2 \rightarrow (2)$$

from (1) & (2) green's theorem is verified.

7. Evaluate $\oint_C 2y dx + 3x^2 dy$ by using Green's theorem where 'C' is the boundary of the region formed by $y=2x$, $y=x^2$



Sol: Given curves $y=2x$, $y=x^2$

$$x^2 = 2x$$

$$x^2 - 2x = 0$$

$$x(x-2) = 0$$

$$x=0, x=2$$

$$\oint_C 2y dx + 3x^2 dy$$

$$M = 2y$$

$$N = 3x^2$$

$$\frac{\partial M}{\partial y} = 2$$

$$\frac{\partial N}{\partial x} = 6x$$

x varies from 0 to 2

y varies from x^2 to $2x$

$$\oint_C 2y dx + 3x^2 dy = \int_0^2 \int_{x^2}^{2x} (6x - 2) dx dy$$

$$= 2 \int_0^2 \left(\int_{x^2}^{2x} (3x - 1) dy \right) dx$$

$$= 2 \int_0^2 \left(\frac{3xy}{2} - y \right)_{x^2}^{2x} dx$$

$$= 2 \int_0^2 (3x(2x) - 2x - (3x x^2 - x^2)) dx$$

$$= 2 \int_0^2 (6x^2 - 2x - 3x^3 + x^2) dx$$

$$= 2 \int_0^2 (3x^2 - x) dx$$

$$= 2 \left(\frac{3x^3}{3} - \frac{x^2}{2} \right)_0^2$$

$$= 2 \left(2^3 - \frac{2^2}{2} \right)$$

$$= 16 - 4$$

$$= 12 //$$

$$= 2 \int_0^2 (-3x^3 + 4x^2 - 2x) dx$$

$$= 2 \left(-3 \frac{x^4}{4} + \frac{4x^3}{3} - 2 \frac{x^2}{2} \right)_0^2$$

$$= \frac{-3}{4} 2^4 + \frac{4}{3} 2^3 - 2 2^2$$

$$= \frac{-3}{2} (16) + \frac{14}{3} (8) - 8$$

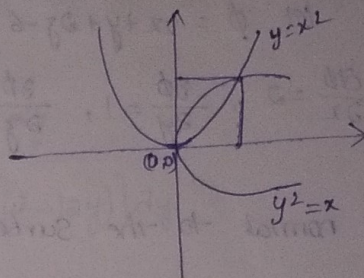
$$= -24 + \frac{112}{3} - 8$$

$$\oint_C 2y dx + 3x^2 dy = \frac{-32 + 112}{3} \\ = \frac{-96 + 112}{3} \\ = \frac{16}{3}$$

8. verify green's theorem in the plane $\oint_C (3x^2 - 8y^2) dx + (4y - 6xy) dy$ where 'C' is the boundary of the region defined by $y = \sqrt{x}$, $y = x^2$.

Sol:- Given 'C' is formed by the curves

$$y = \sqrt{x} \quad , \quad y = x^2 \\ y^2 = x \rightarrow (1), \quad y = x^2 \rightarrow (2)$$



Surface Integrals :- If ' $\vec{F}$ ' is a vector point function define on the surface ' S '. The surface integral of ' $\vec{F}$ ' on ' S ' is denoted by $\int_S \vec{F} \cdot \vec{n} ds$ (or) $\iint_S \vec{F} \cdot \vec{n} ds$, where ' $\vec{n}$ ' is outward drawn unit normal vector to the surface. For evaluation of surface integrals, they can be converted into double integrals by projecting the surface on to any one of the coordinate planes xy, yz, zx .

* If ' S ' is projected on xy -plane. Then $ds = \frac{dxdy}{|\vec{n} \cdot \vec{k}|}$, where $\vec{k}$ is a unit vector along z -axis.

$$\therefore \int_S \vec{F} \cdot \vec{n} ds = \iint_{xy} \vec{F} \cdot \vec{n} \frac{dxdy}{|\vec{n} \cdot \vec{k}|}$$

* If ' S ' is projected on yz -plane then

$$ds = \frac{dydz}{|\vec{n} \cdot \vec{i}|}, \text{ where } \vec{i} \text{ is a unit vector along } x\text{-axis.}$$

$$\therefore \int_S \vec{F} \cdot \vec{n} ds = \iint_{yz} \vec{F} \cdot \vec{n} \frac{dydz}{|\vec{n} \cdot \vec{i}|}$$

* If ' S ' is projected on zx -plane then

$$ds = \frac{dzdx}{|\vec{n} \cdot \vec{j}|} \text{ where } \vec{j} \text{ is a unit vector along } y\text{-axis.}$$

$$\therefore \int_S \vec{F} \cdot d\vec{r} ds = \int \int_S \vec{F} \cdot \vec{n} \frac{dy dz}{|\vec{n} \cdot \vec{j}|}$$

1. Evaluate integral $\oint_S \vec{F} \cdot \vec{n} ds$ where 'S' is the surface of the plane $2x+y+2z=6$ in the first octant and $\vec{F}=(x+y^2)\vec{i}-2xz\vec{j}+2yz\vec{k}$

Sol:- Given $\vec{F}=(x+y^2)\vec{i}-2xz\vec{j}+2yz\vec{k}$

Let 'S' is the surface of the plane

$2x+y+2z=6$ in the first octant. (xyz is +ve)

Let $\phi = 2x+y+2z-6$ then $\nabla\phi = \vec{i} \frac{\partial\phi}{\partial x} + \vec{j} \frac{\partial\phi}{\partial y} + \vec{k} \frac{\partial\phi}{\partial z}$

$\frac{\partial\phi}{\partial x} = 2, \frac{\partial\phi}{\partial y} = 1, \frac{\partial\phi}{\partial z} = 2 \Rightarrow \nabla\phi = 2\vec{i} + \vec{j} + 2\vec{k}$ is the normal to the surface

Unit normal to the surface $\vec{n} = \frac{\nabla\phi}{|\nabla\phi|}$

$$= \frac{2\vec{i} + \vec{j} + 2\vec{k}}{\sqrt{4+1+4}}$$

$$= \frac{2\vec{i} + \vec{j} + 2\vec{k}}{3}$$

$$\text{Now } \vec{F} \cdot \vec{n} = ((x+y^2)\vec{i} - 2xz\vec{j} + 2yz\vec{k}) \cdot \frac{2\vec{i} + \vec{j} + 2\vec{k}}{3}$$

$$= \frac{2(x+y^2) - 2xz + 4yz}{3}$$

$$= \frac{2x + 2y^2 - 2xz + 4yz}{3}$$

$$\vec{F} \cdot \vec{n} = \frac{2y^2 + 4yz}{3}$$

Let 'r' be the projection of 'x' on yz-plane. Then

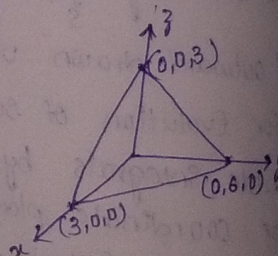
$$ds = \frac{dy dz}{|\vec{n} \cdot \vec{j}|} \Rightarrow |\vec{n} \cdot \vec{j}| = \left(\frac{2\vec{i} + \vec{j} + 2\vec{k}}{3} \cdot \vec{j} \right)$$

$$= \frac{dy dz}{3}$$

$$= \frac{dy dz}{2/3}$$

In yz-plane, $x=0$

$$\Rightarrow y + 2z = 6$$



in y-axis, $z=0 \rightarrow y$ varies from 0 to 6
 & z varies from 0 to $\frac{6-y}{2}$

$$\begin{aligned}
 \therefore \int_S \vec{F} \cdot \vec{n} \, ds &= \int_0^6 \int_0^{\frac{6-y}{2}} \frac{1}{8} (2y^2 + 4yz) \, dz \, dy \cdot \frac{8}{2} \\
 &= \frac{1}{2} \int_0^6 \int_0^{\frac{6-y}{2}} (2y^2 + 4yz) \, dz \, dy \\
 &= \frac{1}{2} \int_0^6 \left(2y^2 z + 2y \frac{z^2}{2} \right) \Big|_0^{\frac{6-y}{2}} dy \\
 &= \frac{1}{2} \int_0^6 \left(2y^2 \left(\frac{6-y}{2} \right) + 2y \left(\frac{6-y}{2} \right)^2 \right) dy \\
 &= \frac{1}{2} \int_0^6 \left(6y^2 - y^3 + \frac{y}{2} (36 + y^2 - 12y) \right) dy \\
 &= \frac{1}{2} \int_0^6 \left(6y^2 - y^3 + \frac{36y}{2} + \frac{y^3}{2} - \frac{12y^2}{2} \right) dy \\
 &= \frac{1}{2} \int_0^6 \left(\frac{12y^2}{2} - 2y^3 + 36y + \frac{y^3}{2} - 12y^2 \right) dy \\
 &= \frac{1}{2} \int_0^6 (-y^3 + 36y) \, dy \\
 &= \frac{1}{4} \left(-\frac{y^4}{4} + 36 \frac{y^2}{2} \right) \Big|_0^6 \\
 &= \frac{1}{4} \left(\frac{-1296}{4} + 18(36) \right) \\
 &= \frac{1}{4} \left(\frac{-1296}{4} + 648 \right) \\
 &= \frac{1}{4} \left(\frac{-1296 + 2592}{4} \right) \\
 &= \frac{1296}{16} \\
 &= 81 \quad //
 \end{aligned}$$

2. Find $\int_S \vec{F} \cdot \vec{n} \, ds$ where $\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$ and 'S' is the surface of the sphere $x^2 + y^2 + z^2 = 9$ in the first octant.

Sol:- Given $\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$
 'S' is the surface of the sphere
 $x^2 + y^2 + z^2 = 9$

Let $\phi = x^2 + y^2 + z^2 - 9$

$\frac{\partial \phi}{\partial x} = 2x$, $\frac{\partial \phi}{\partial y} = 2y$, $\frac{\partial \phi}{\partial z} = 2z$

$\nabla \phi = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$

is the normal to the surface

$\therefore$ unit normal to the surface $\vec{n} = \frac{\nabla \phi}{|\nabla \phi|}$

$= \frac{2x\vec{i} + 2y\vec{j} + 2z\vec{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}}$

$= \frac{2(x\vec{i} + y\vec{j} + z\vec{k})}{2\sqrt{x^2 + y^2 + z^2}}$

$= \frac{x\vec{i} + y\vec{j} + z\vec{k}}{\sqrt{9}}$

$= \frac{x\vec{i} + y\vec{j} + z\vec{k}}{3}$

Now $\vec{F} \cdot \vec{n} = (yz\vec{i} + zx\vec{j} + xy\vec{k}) \cdot \frac{x\vec{i} + y\vec{j} + z\vec{k}}{3}$

$= \frac{1}{3} (xyz + xyz + xyz)$

$= \frac{3xyz}{3} \Rightarrow xyz$

Let 'R' be the projection of 'S' surface 'S' on xy-plane

$ds = \frac{dxdy}{|\vec{n} \cdot \vec{k}|} \Rightarrow |\vec{n} \cdot \vec{k}| = \left| \frac{x\vec{i} + y\vec{j} + z\vec{k}}{3} \cdot \vec{k} \right|$

$= \left| \frac{z}{3} \right|$

$ds = \frac{3}{z} dxdy$

Limits :- The region R in xy-plane

$z \geq 0$, $x^2 + y^2 = 9$

$y^2 = 9 - x^2$

$y = \sqrt{9 - x^2}$

$\therefore$ y varies from 0 to $\sqrt{9 - x^2}$

in x-axis, $y = 0$, then x varies from 0 to 3.

$$\begin{aligned}
 \therefore \int_S \vec{F} \cdot \vec{n} \, ds &= \int_0^3 \int_0^{\sqrt{9-x^2}} 2y \cdot \frac{3}{2} \, dy \, dx \\
 &= 3 \int_0^3 \left[\int_0^{\sqrt{9-x^2}} xy \, dy \right] dx \\
 &= 3 \int_0^3 \left[x \frac{y^2}{2} \right]_0^{\sqrt{9-x^2}} dx \\
 &= \frac{3x}{2} \int_0^3 (\sqrt{9-x^2})^2 dx \\
 &= \frac{3x}{2} \int_0^3 (9-x^2) dx \\
 &= \frac{3}{2} \int_0^3 x(9-x^2) dx \\
 &= \frac{3}{2} \int_0^3 (9x - x^3) dx \\
 &= \frac{3}{2} \left[\frac{9x^2}{2} - \frac{x^4}{4} \right]_0^3 \\
 &= \frac{3}{2} \left(\frac{9}{2} (3)^2 - \frac{1}{4} (3)^4 \right) \\
 &= \frac{3}{2} \left(\frac{81}{2} - \frac{81}{4} \right) \\
 &= \frac{3}{2} \left(\frac{162-81}{4} \right) \\
 &= \frac{3}{8} (81) \Rightarrow \frac{243}{8}
 \end{aligned}$$

3. Evaluate $\int_S \vec{F} \cdot \vec{n} \, ds$ where $\vec{F} = z\vec{i} + x\vec{j} - 3y^2z\vec{k}$ and 's' is the surface of the cylinder $x^2 + y^2 = 16$ included in the first Octant b/w $z=0$ & $z=5$

Sol:- Given $\vec{F} = z\vec{i} + x\vec{j} - 3y^2z\vec{k}$
's' is the surface of the cylinder

$$x^2 + y^2 = 16$$

$$\text{let } \phi = x^2 + y^2 - 16$$

$$\frac{\partial \phi}{\partial x} = 2x \quad \frac{\partial \phi}{\partial y} = 2y \quad , \quad \frac{\partial \phi}{\partial z} = 0$$

$\nabla \phi = 2x\vec{i} + 2y\vec{j}$ is the normal to the surface 's'

unit normal vector to the surface $\vec{n} = \frac{\nabla \phi}{|\nabla \phi|}$

$$\begin{aligned}\bar{n} &= \frac{2x\bar{i} + 2y\bar{j}}{\sqrt{4x^2 + 4y^2}} \\ &= \frac{2(x\bar{i} + y\bar{j})}{2\sqrt{x^2 + y^2}} \\ &= \frac{x\bar{i} + y\bar{j}}{\sqrt{16}} \\ \bar{n} &= \frac{x\bar{i} + y\bar{j}}{4}\end{aligned}$$

$$\begin{aligned}\text{Now } \bar{F} \cdot \bar{n} &= (z\bar{i} + x\bar{j} - 3y^2\bar{k}) \cdot \frac{x\bar{i} + y\bar{j}}{4} \\ &= \frac{xyz\bar{i} + xy\bar{j}}{4}\end{aligned}$$

The curved surface 's' is $\perp^{\text{r}}$ to xy-plane so we cannot project in xy-plane.

Let 'R' be the projection of 's' on yz-plane then

$$ds = \frac{dydz}{|\bar{n} \cdot \bar{i}|} \Rightarrow |\bar{n} \cdot \bar{i}| = \left| \frac{x\bar{i} + y\bar{j}}{4} \cdot \bar{i} \right|$$

$$= \left| \frac{x}{4} \right|$$

$$ds = \frac{4}{x} dydz$$

$$\therefore \int_S \bar{F} \cdot \bar{n} ds = \int \int_R \frac{1}{x} (xyz\bar{i} + xy\bar{j}) \cdot \frac{4}{x} dydz$$

$$= \int \int_R (z + y) dydz$$

z varies from 0 to 5

y varies from 0 to 4

$$\int_S \bar{F} \cdot \bar{n} ds = \int_{y=0}^4 \int_{z=0}^5 (z + y) dydz$$

$$= \int_0^5 \left[zy + \frac{y^2}{2} \right]_0^4 dz$$

$$= \int_0^5 \left(3u + \frac{16}{2} \right) dz$$

$$= 4 \int_0^5 (z + 2) dz$$

$$= 4 \int_0^5 \left(\frac{z^2}{2} + 2z \right) dz$$

$$= 4 \left(\frac{5^2}{2} + 2(5) \right)$$

$$= 4 \left(\frac{25}{2} + 10 \right)$$

$$= 2 \left(\frac{25 + 20}{2} \right)$$

$$= 2(45)$$

$$= 90 //$$

4. Evaluate $\iint_S \vec{A} \cdot \vec{n} \, ds$ over the entire surface 'S' of the region bounded by the cylinder $x^2 + z^2 = 9$, $x=0$, $y=0$, $z=0$ and $y=8$.

Where $\vec{A} = 6z\vec{i} + (2x+y)\vec{j} + x\vec{k}$

Sol:- The surface 'S' consist of '5' surface 's'

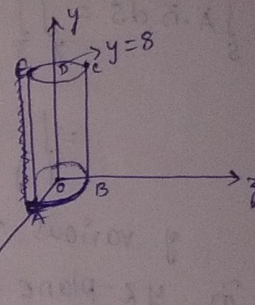
namely $S_1 = ABCE$

$S_4 = AODE$

$S_2 = AOB$

$S_5 = OBCD$

$S_3 = EDC$



$$\therefore \iint_S \vec{A} \cdot \vec{n} \, ds = \int_{S_1} \vec{A} \cdot \vec{n} \, ds + \int_{S_2} \vec{A} \cdot \vec{n} \, ds + \int_{S_3} \vec{A} \cdot \vec{n} \, ds + \int_{S_4} \vec{A} \cdot \vec{n} \, ds + \int_{S_5} \vec{A} \cdot \vec{n} \, ds$$

$$\text{let } \phi = x^2 + z^2 - 9$$

$$\frac{\partial \phi}{\partial x} = 2x, \quad \frac{\partial \phi}{\partial y} = 0, \quad \frac{\partial \phi}{\partial z} = 2z$$

$\nabla \phi = 2x\vec{i} + 2z\vec{k}$ is the normal vect to the surface

$$\therefore \text{unit normal to the surface } \vec{n} = \frac{\nabla \phi}{|\nabla \phi|}$$

$$= \frac{2x\vec{i} + 2z\vec{k}}{\sqrt{4x^2 + 4z^2}}$$

$$= \frac{2(x\vec{i} + z\vec{k})}{2\sqrt{x^2 + z^2}}$$

$$= \frac{x\vec{i} + z\vec{k}}{\sqrt{9}}$$

$$\vec{n} = \frac{x\vec{i} + z\vec{k}}{3}$$

$$\begin{aligned}\text{Now } \vec{A} \cdot \vec{n} &= (6z\vec{i} + (2x+y)\vec{j} - x\vec{k}) \cdot \frac{x\vec{i} + y\vec{j}}{3} \\ &= \frac{1}{3}(6xz - xz) \\ &= \frac{1}{3}(5xz)\end{aligned}$$

Consider 'R' be the projection of 'S' on yz-plane

$$ds = \frac{dydz}{|\vec{n} \cdot \vec{i}|} \Rightarrow |\vec{n} \cdot \vec{i}| = \left| \frac{x\vec{i} + y\vec{j}}{3} \cdot \vec{i} \right| = \left| \frac{x}{3} \right|$$

$$ds = \frac{3}{x} dydz$$

$$\begin{aligned}\therefore \int_S \vec{A} \cdot \vec{n} ds &= \int \int_R \frac{5xz}{3} \cdot \frac{3}{x} dydz \\ &= \int \int_R 5z dydz\end{aligned}$$

y varies from 0 to 8

In yz-plane, $x=0$ at $z^2=9$
 $z=3$

z varies from 0 to 3

$$\therefore \int_S \vec{A} \cdot \vec{n} ds = \int_{y=0}^8 \int_0^3 5z dydz$$

$$= 5 \int_0^3 z (y^2)_0^8 dz$$

$$= 5 \int_0^3 8z dz$$

$$= 40 \left[\frac{z^2}{2} \right]_0^3$$

$$= \frac{40}{2} (3^2) \Rightarrow 20(9)$$

$$= 180 //$$

Case (ii): - on the surface AOB :-

This surface lies on xz-plane and the eqn of the surface is $y=0$

The outward unit normal vector is along '-ve' y

$$\Rightarrow \vec{n} = -\vec{j}$$

$$\vec{A} \cdot \vec{n} = (6\vec{j} + (2x+y)\vec{j} - x\vec{k}) \cdot (-\vec{j})$$

$$= -(2x+y)$$

$$= -2x - y$$

$$= -2x$$

$$\because y=0$$

OAB is a circle, $x^2 + y^2 = 9$

$$\Rightarrow z^2 = 9 - x^2$$

$$z = \pm \sqrt{9 - x^2}$$

z varies from $-\sqrt{9-x^2}$ to $\sqrt{9-x^2}$

to set x limits, put $z=0$, $x^2=9$

$$x = \pm 3$$

$\therefore x$

x varies from -3 to 3 .

$$\int \vec{A} \cdot \vec{n} \, ds = \int_{x=-3}^3 \int_{z=-\sqrt{9-x^2}}^{\sqrt{9-x^2}} -2x \frac{dx dz}{|\vec{n} \cdot \vec{j}|}$$

$$= -2 \int_{x=-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} x \frac{dx dz}{|-1|}$$

$$= -2 \int_{x=-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} x \, dx \, dz$$

$$= -2 \int_{-3}^3 x \left[z \right]_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dx$$

$$= -2 \int_{-3}^3 x \left(\sqrt{9-x^2} + \sqrt{9-x^2} \right) dx$$

$$= -2 \int_{-3}^3 2x \sqrt{9-x^2} \, dx$$

$$= -4 \int_{-3}^3 x \sqrt{9-x^2} \, dx$$

$$= 0$$

$$\begin{cases} \int_{-a}^a f(x) \, dx = 0, \text{ where } f(x) \text{ is odd} \\ f(-x) = -f(x) \text{ if } f \text{ is odd} \end{cases}$$

$$\therefore \int_{-3}^3 -x \sqrt{9-x^2} \, dx = - \int_{-3}^3 x \sqrt{9-x^2} \, dx$$

Case (ii): on the surface EDC:

This surface is parallel to xz-plane, on the eqn

of the surface is $y=8$.

and the outward unit normal vector along '+ve' y-axis
 $\Rightarrow \vec{n} = \vec{j}$

$$\begin{aligned}\vec{A} \cdot \vec{n} &= 2x + y \\ &= 2x + 8 \quad \because y = 8\end{aligned}$$

consider the projection on xz-plane

$$\begin{aligned}\text{So } ds &= \frac{dx dz}{|\vec{n} \cdot \vec{j}|} \\ &= dx dz\end{aligned}$$

as it is a circle in xz-plane.

$$\text{i.e.; } x^2 + z^2 = 9$$

$$z^2 = 9 - x^2 \Rightarrow z = \pm \sqrt{9 - x^2}$$

$\therefore z$ varies from $-\sqrt{9 - x^2}$ to $\sqrt{9 - x^2}$

x limits put $z = 0$, $x^2 = 9$
 $x = \pm 3$

x varies from -3 to 3

$$\begin{aligned}\int_S \vec{A} \cdot \vec{n} \, ds &= \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (2x + 8) \, dx \, dz \\ &= \int_{-3}^3 \left[2xz + 8z \right]_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dx \\ &= \int_{-3}^3 \left(2x\sqrt{9-x^2} + 8\sqrt{9-x^2} - (-2x\sqrt{9-x^2} - 8\sqrt{9-x^2}) \right) dx \\ &= \int_{-3}^3 \left(2x\sqrt{9-x^2} + 8\sqrt{9-x^2} + 2x\sqrt{9-x^2} + 8\sqrt{9-x^2} \right) dx \\ &= \int_{-3}^3 \left(4x\sqrt{9-x^2} + 16\sqrt{9-x^2} \right) dx \\ &= 2 \int_{-3}^3 2x\sqrt{9-x^2} \, dx + 16 \int_{-3}^3 \sqrt{9-x^2} \, dx \\ &= 2(0) + 16(2) \int_0^3 \sqrt{9-x^2} \, dx \\ &= 32 \int_0^3 \sqrt{3^2 - x^2} \, dx \\ &= 32 \left[\frac{x}{2} \sqrt{3^2 - x^2} + \frac{9}{2} \sin^{-1} \left(\frac{x}{3} \right) \right]_0^3\end{aligned}$$

$$\begin{aligned}
 &= 32 \left(\frac{3}{2} \sqrt{9-3^2} + \frac{9}{2} \sin^{-1}\left(\frac{3}{3}\right) - 0 + \sin^{-1}(0) \right) \\
 &= \frac{32 \times 16}{2} 3\sqrt{9-3^2} + \frac{32}{2} 9 \sin^{-1}(1) \\
 &= 16(3)\sqrt{9-3^2} + 16(9) \sin^{-1}(1) \\
 &= 0 + \frac{72}{144} \left(\frac{\pi}{2} \right) \\
 &= 72 \pi
 \end{aligned}$$

Case (iv):- This surface is on the surface 'AODE'. $z=0$.

This surface lies on xy-plane

$$\Rightarrow \vec{n} = -\vec{k}$$

$$\vec{A} \cdot \vec{n} = x$$

consider the projection on xy-plane

$$ds = \frac{dx dy}{|\vec{n} \cdot \vec{k}|}$$

$$ds = \frac{dx dy}{1} \Rightarrow dx dy$$

Limits:- y varies from 0 to 8

In xy-plane $z=0$, i.e. $x^2 = 9$
 $x = \pm 3$

x varies from -3 to 3

$$\begin{aligned}
 \int_S \vec{A} \cdot \vec{n} ds &= \int_{x=-3}^3 \int_{y=0}^8 x dx dy \Rightarrow \int_0^8 \left(\int_{-3}^3 x dx \right) dy \\
 &\Rightarrow \int_0^8 (0) dy \Rightarrow 0
 \end{aligned}$$

$$\begin{aligned}
 &= \int_0^8 \left[\frac{x^2}{2} \right]_{-3}^3 dy \\
 &= \int_0^8 \left(\frac{9}{2} + \frac{9}{2} \right) dy \Rightarrow 9 \int_0^8 dy \Rightarrow 9[y]_0^8 \\
 &\Rightarrow 9(8)
 \end{aligned}$$

Case (v):- on the surface OBCD:-

This surface lies on xz-plane and the eqn of

the surface is $x=0$.

$$\vec{n} = -\vec{i}$$

$$\vec{A} \cdot \vec{n} = -6z$$

$$\Rightarrow \text{xz-plane} \Rightarrow ds = \frac{dx dz}{|\vec{n} \cdot \vec{i}|} \Rightarrow dx dz$$

limits :- y varies from 0 to 8
 in yz -plane, $x=0$, i.e. $0+z^2=9$
 $z=\pm 3$

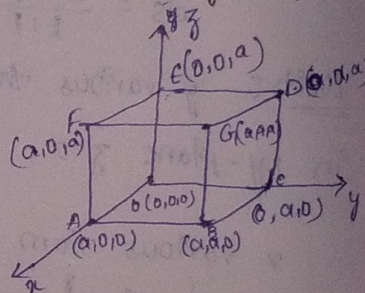
z varies from 0 to 3

$$\begin{aligned}\int_S \vec{A} \cdot \vec{n} \, ds &= \int_{x=0}^3 \int_{y=0}^8 -6z \, dy \, dz \\ &= -6 \int_0^8 \left(\int_0^3 z \, dz \right) dy \\ &= -6 \int_0^8 \left[\frac{z^2}{2} \right]_0^3 dy \\ &= -6 \int_0^8 \frac{9}{2} dy \\ &= -27 [y]_0^8 \Rightarrow -27(8) \Rightarrow -216\end{aligned}$$

Ex 5.11 If $\vec{F} = xz\vec{i} - y^2\vec{j} + yz\vec{k}$ evaluate $\int_S \vec{F} \cdot \vec{n} \, ds$ where 'S' is the surface of the cube bounded by $x=0, x=a, y=0, y=a, z=0, z=a$

Sol:- Given $\vec{F} = xz\vec{i} - y^2\vec{j} + yz\vec{k}$
 the surface 'S' consist of 6 surfaces

$$\begin{aligned}S_1 &= OABC & S_4 &= OCDE \\ S_2 &= EFGD & S_5 &= BCDG \\ S_3 &= ABGF & S_6 &= ADEF\end{aligned}$$



$$\int_S \vec{F} \cdot \vec{n} \, ds = \int_{S_1} \vec{F} \cdot \vec{n} \, ds + \int_{S_2} \vec{F} \cdot \vec{n} \, ds + \int_{S_3} \vec{F} \cdot \vec{n} \, ds + \int_{S_4} \vec{F} \cdot \vec{n} \, ds + \int_{S_5} \vec{F} \cdot \vec{n} \, ds + \int_{S_6} \vec{F} \cdot \vec{n} \, ds$$

Case (i) :- 'on S_1 ' :-

This surface lies on xy -plane then $z=0$ and the eqn of surface is $z=0$

$$\Rightarrow \vec{n} = -\vec{k}$$

$$\begin{aligned}\text{Now } \vec{F} \cdot \vec{n} &= (xz\vec{i} - y^2\vec{j} + yz\vec{k}) \cdot (-\vec{k}) \\ &= -yz \Rightarrow -y(0) \Rightarrow 0\end{aligned}$$

$$ds = \frac{dx \, dy}{|\vec{n} \cdot \vec{k}|} \Rightarrow \frac{dx \, dy}{|-1|} \Rightarrow dx \, dy$$

x varies from 0 to a
 y varies from 0 to a

$$\int_S \vec{F} \cdot \vec{n} ds = \int_0^a \int_0^a 0 \, dx \, dy$$

$$= 0$$

case (ii) on S_2 :-

eqn of the surface is $z = a$
 $\Rightarrow \vec{n} = \vec{k}$

$$\vec{F} \cdot \vec{n} = yz \quad \because z = a$$

$$= ya$$

$$ds = dx \, dy$$

The surface S_2 can be projected on xy -plane

$$ds = \frac{dx \, dy}{|\vec{n} \cdot \vec{k}|}$$

$$ds = dx \, dy$$

and x varies from 0 to a
 y varies from 0 to a

$$\int_S \vec{F} \cdot \vec{n} \, ds = \int_0^a \int_0^a ay \, dx \, dy$$

$$= a \int_0^a \left[\frac{y^2}{2} \right]_0^a \, dy$$

$$= a \int_0^a \frac{a^2}{2} \, dy$$

$$= \frac{a^3}{2} [y]_0^a \Rightarrow \frac{a^3}{2} (a) \Rightarrow \frac{a^4}{2}$$

case (iii) on S_3 :-

The eqn of the surface is $x = a$
 $\vec{n} = \vec{i}$

$$\vec{F} \cdot \vec{n} = uxz$$

$$= uax$$

$\Rightarrow$ The surface S_3 can be projected on yz -plane

$$ds = \frac{dydz}{|\vec{n} \cdot \vec{i}|}$$

$$ds = dydz$$

y varies from 0 to a
z varies from 0 to a

$$\int_S \vec{F} \cdot \vec{n} ds = \int_0^a \int_0^a uaz dydz$$

$$= ua \int_0^a \left(\frac{z^2}{2} \right)_0^a dy$$

$$= ua \int_0^a \frac{a^2}{2} dy \Rightarrow 2a^3 \int_0^a y$$

$$\Rightarrow 2a^3(a) \Rightarrow 2a^4$$

case (iv) on S_u :- This surface lies on yz-plane

The eqn of the surface is $x=0$

$$\vec{n} = -\vec{i}$$

$$\vec{F} \cdot \vec{n} = -uxz$$

$$= -u(0) \Rightarrow 0$$

z varies from 0 to a

y varies from 0 to a

$$\int_S \vec{F} \cdot \vec{n} ds = \int_0^a \int_0^a 0 dydz$$

$$= 0$$

case (v) on S_5 :-

The eqn of the surface is $y=a$

$$\vec{n} = \vec{j}$$

$$\vec{F} \cdot \vec{n} = -y^2$$

$$= -a^2$$

The surface S_5 can be projected on xz-plane

$$ds = \frac{dxdz}{|\vec{n} \cdot \vec{j}|}$$

$$ds = dxdz$$

and x varies from 0 to a

z varies from 0 to a

$$\begin{aligned}
 \int_S \vec{F} \cdot \vec{n} \, ds &= \int_0^a \int_0^a -a^2 \, dx \, dz \\
 &= -a^2 \int_0^a [x]_0^a \, dz \\
 &= -a^2 \int_0^a a \, dz \Rightarrow -a^3 [z]_0^a \\
 &\Rightarrow -a^3(a) \Rightarrow -a^4
 \end{aligned}$$

case (iv) on S_6 :-

The surface lies on yz -plane

The eqn of the surface is $y=0$

$$\vec{n} = -\vec{j}$$

$$\vec{F} \cdot \vec{n} = y^2 = 0 \quad ds = \frac{dx \, dz}{|\vec{n} \cdot \vec{j}|} = \frac{dx \, dz}{1} = dx \, dz$$

x varies from 0 to a

z varies from 0 to a

$$\begin{aligned}
 \int_S \vec{F} \cdot \vec{n} \, ds &= \int_0^a \int_0^a 0 \, dx \, dz \\
 &= 0
 \end{aligned}$$

from eqn (1)

$$\begin{aligned}
 \int_S \vec{F} \cdot \vec{n} \, ds &= 0 + \frac{a^4}{2} + 2a^4 + 0 - a^4 + 0 \\
 &= \frac{a^4}{2} + a^4
 \end{aligned}$$

$$\int_S \vec{F} \cdot \vec{n} \, ds = \frac{3a^4}{2} //$$

Physical Interpretation of Surface Integral :-

If ' $\vec{F}$ ' represents the velocity of a fluid fluid at any point on the closed surface ' S '. Then surface integral

$\int_S \vec{F} \cdot \vec{n} \, ds$ represents the rate of flow of fluid through the surface ' S ' and the surface integral is called flux of $\vec{F}$.

Find the flux of surface field $\vec{A} = (x-2z)\vec{i} + (x+3y+z)\vec{j} + (5x+y)\vec{k}$ through upper side of the triangle ABC with vertices $A(1,0,0)$, $B(0,1,0)$, $C(0,0,1)$

Volume integrals :- If 'v' is the volume bounded by the surface 's'. Then the $\iiint_V \phi \, dv$ (or) $\iiint_V \vec{F} \, dv$ are called as volume integrals. Where ϕ is a scalar function and $\vec{F}$ is a vector function.

If $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ then the volume integral

$$\begin{aligned}\iiint_V \vec{F} \, dv &= \iiint_V (F_1\vec{i} + F_2\vec{j} + F_3\vec{k}) \, dx \, dy \, dz \\ &= \vec{i} \iiint_V F_1 \, dx \, dy \, dz + \vec{j} \iiint_V F_2 \, dx \, dy \, dz + \vec{k} \iiint_V F_3 \, dx \, dy \, dz\end{aligned}$$

1. If $\vec{F} = (2x^2-3z)\vec{i} - 2xy\vec{j} - ux\vec{k}$ then find (i) $\int_V \text{div } \vec{F} \, dv$

(ii) $\int_V \text{curl } \vec{F} \, dv$. Where 'v' is the ^{closed} region bounded by the planes $x=0$, $y=0$, $z=0$ and $2x+2y+z=4$.

Sol: (i) Given $\vec{F} = (2x^2-3z)\vec{i} - 2xy\vec{j} - ux\vec{k}$

$$\begin{aligned}\text{Now } \text{div } \vec{F} &= \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \\ &= \frac{\partial}{\partial x} (2x^2-3z) + \frac{\partial}{\partial y} (-2xy) + \frac{\partial}{\partial z} (-ux) \\ &= 4x - 2x + 0 \\ &= 2x\end{aligned}$$

To find $\int_V \text{div } \vec{F} \, dv$:-

Here 'v' is the volume bounded by the lines $x=0$, $y=0$, $z=0$ and $2x+2y+z=4$

From the given eqn

$$z = 4 - 2x - 2y$$

'z' varies from 0 to '4-2x-2y'.

xy-plane ($z=0$), $2x+2y=4$

$$\Rightarrow x+y=2$$

$$y = 2-x$$

y varies from 0 to $2-x$
 to set the x limits, put $y=z=0$
 $2x+0+0=4$
 $x=2$

$\therefore x$ varies from 0 to 2

$$\begin{aligned}
 \int_V \text{div } \vec{F} \, dV &= \int_{x=0}^2 \int_{y=0}^{2-x} \int_{z=0}^{4-2x-2y} 2x \, dz \, dy \, dx \\
 &= \int_{x=0}^2 \int_{y=0}^{2-x} 2x \left(\int_{z=0}^{4-2x-2y} dz \right) dy \, dx \\
 &= \int_0^2 \int_0^{2-x} 2x \left(z \right)_0^{4-2x-2y} dy \, dx \\
 &= \int_0^2 \int_0^{2-x} 2x (4-2x-2y) dy \, dx \\
 &= 2 \int_0^2 \int_0^{2-x} (4x-2x^2-2xy) dy \, dx \\
 &= \int_0^2 \left(8x \left(y \right)_0^{2-x} - 4x^2 \left(y \right)_0^{2-x} - 2x \left(\frac{y^2}{2} \right)_0^{2-x} \right) dx \\
 &= \int_0^2 \left(8x(2-x) - 4x^2(2-x) - 2x(2-x)^2 \right) dx \\
 &= \int_0^2 \left(16x - 8x^2 - 8x^2 + 4x^3 - 2x(4 - 4x + x^2) \right) dx \\
 &= \int_0^2 \left(16x - 16x^2 + 4x^3 - 8x + 8x^2 - 2x^3 \right) dx \\
 &= \int_0^2 \left(8x - 8x^2 + 2x^3 \right) dx \\
 &= \left[8 \frac{x^2}{2} - 8 \frac{x^3}{3} + 2 \frac{x^4}{4} \right]_0^2 \\
 &= 4(2)^2 - \frac{8}{3}(2)^3 + \frac{(2)^4}{2} \\
 &= 16 - \frac{64}{3} + 8 \\
 &= 24 - \frac{64}{3} \\
 &= \frac{72-64}{3} \\
 &= \frac{8}{3}
 \end{aligned}$$

$$(ii) \text{curl } \vec{F} = \nabla \times \vec{F}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x^2 - 3z & -2xy & -ux \end{vmatrix}$$

$$= \vec{i} \left(\frac{\partial}{\partial y} (-ux) + \frac{\partial}{\partial z} (2xy) \right) - \vec{j} \left(\frac{\partial}{\partial x} (-ux) - \frac{\partial}{\partial z} (2x^2 - 3z) \right) + \vec{k} \left(\frac{\partial}{\partial x} (-2xy) - \frac{\partial}{\partial y} (2x^2 - 3z) \right)$$

$$= \vec{i} (0 + 0) - \vec{j} (-u - (-3)) + \vec{k} (-2y + 0)$$

$$= -\vec{j}(-1) + \vec{k}(-2y)$$

$$\text{curl } \vec{F} = \vec{j} - 2y\vec{k}$$

To find $\int_V \text{curl } \vec{F}$ is The limits same as above

$$\therefore \int_V \text{curl } \vec{F} \, dv = \int_{x=0}^2 \int_{y=0}^{2-x} \int_{z=0}^{u-2x-2y} (\vec{j} - 2y\vec{k}) \, dx \, dy \, dz$$

$$= \int_0^2 \int_0^{2-x} \left(\int_0^{u-2x-2y} \vec{j} \, dz \right) dy \, dx - 2\vec{k} \int_0^2 \int_0^{2-x} \left(\int_0^{u-2x-2y} y \, dz \right) dy \, dx$$

$$= \vec{j} \int_0^2 \int_0^{2-x} \left[z \right]_0^{u-2x-2y} dy \, dx - 2\vec{k} \int_0^2 \int_0^{2-x} y \left[z \right]_0^{u-2x-2y} dy \, dx$$

$$= \vec{j} \int_0^2 \int_0^{2-x} (u-2x-2y) dy \, dx - 2\vec{k} \int_0^2 \int_0^{2-x} y(u-2x-2y) dy \, dx$$

$$= \vec{j} \int_0^2 \left[uy - 2xy - 2 \frac{y^2}{2} \right]_0^{2-x} dx - 2\vec{k} \int_0^2 \int_0^{2-x} (uy - 2xy - 2y^2) dy \, dx$$

$$= \vec{j} \int_0^2 (u(2-x) - 2x(2-x) - (2-x)^2) dx - 2\vec{k} \int_0^2 \left[\frac{ux^2y^2}{2} - 2x \frac{y^3}{3} - 2 \frac{y^4}{4} \right]_0^{2-x} dx$$

$$= \vec{j} \int_0^2 (8 - ux - ux^2 - 2x^2 - (u + x^2 - 2x^2)) dx - 2\vec{k} \int_0^2 \left(2(2-x)^2 - x(2-x)^2 - \frac{2}{3} (2-x)^3 \right) dx$$

$$= \vec{j} \int_0^2 (8 - 8x - 2x^2 - u - x^2 + ux) dx - 2\vec{k} \int_0^2 \left(2(u + x^2 - ux) - x(u + x^2 - ux) - \frac{2}{3} (2-x)^3 \right) dx$$

$$\begin{aligned}
&= \bar{j} \int_0^2 (4 - ux - 3x^2) dx - 2\bar{k} \int_0^2 (8 + 2x^2 - 8x - ux - x^3 + ux^2 - \frac{2}{3}(2-x)^3) dx \\
&= \bar{j} \left[4x - \frac{ux^2}{2} - \frac{3x^3}{3} \right]_0^2 - 2\bar{k} \left[8x + \frac{2x^3}{3} - \frac{8x^2}{2} - \frac{ux^2}{2} - \frac{x^4}{4} + \frac{ux^3}{3} - \frac{2}{3} \left(\frac{(2-x)^4}{4} (1) \right) \right]_0^2 \\
&= \bar{j} (4(2) - \frac{u(2)^2}{2} - (2)^3) - 2\bar{k} \left[2(8) + \frac{2}{3}(2)^3 - \frac{8}{2}(2)^2 - \frac{u(2)^2}{2} - \frac{2^4}{4} + \frac{u(2)^3}{3} - \frac{2}{3} \left(\frac{(2-2)^4}{4} - \frac{(2-0)^4}{4} \right) \right] \\
&= \bar{j} (8 - 8 - 8) - 2\bar{k} \left(16 + \frac{2}{3}(8) - 16 - 8 - u + \frac{u}{3}(8) + \frac{1}{6}(0) \right) \\
&= \bar{j} (-8) - 2\bar{k} \left(\frac{16}{3} + \frac{32}{3} - 12 \right) \\
&= -8\bar{j} - 2\bar{k} \left(\frac{16}{3} - 12 \right) \\
&= -8\bar{j} - 2\bar{k}(4) \Rightarrow -8\bar{j} - 8\bar{k} \Rightarrow -8(\bar{j} + \bar{k})
\end{aligned}$$

Gauss Divergent Theorem :- (Relation b/w surface & volume integrals)

If $\vec{F}$ is a vector point function having continuous partial derivatives in the region 'v' bounded by the closed surface 'S', then $\iint_S \vec{F} \cdot \vec{n} ds = \iiint_v \nabla \cdot \vec{F} dv$

1. ~~we~~ Evaluate use divergent theorem, to evaluate $\iint_S \vec{F} \cdot \vec{n} ds$ where $\vec{F} = ux\bar{i} - 2y^2\bar{j} + z^2\bar{k}$ and 'S' is the surface bounded by the region $x^2 + y^2 = 4$, $z=0$ & $z=3$

Sol:- Given $\vec{F} = ux\bar{i} - 2y^2\bar{j} + z^2\bar{k}$
 $\vec{F} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

$$\begin{aligned}
\nabla \cdot \vec{F} &= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \\
&= \frac{\partial}{\partial x}(ux) - \frac{\partial}{\partial y}(2y^2) + \frac{\partial}{\partial z}(z^2) \\
&= u - 4y + 2z
\end{aligned}$$

by divergence theorem

$$\iint_S \vec{F} \cdot \vec{n} ds = \iiint_v (\nabla \cdot \vec{F}) dv$$

here z varies from 0 to 3
 $x^2 + y^2 = 4 \Rightarrow y^2 = 4 - x^2$
 y varies from $-\sqrt{4-x^2}$ to $\sqrt{4-x^2}$

limits of 'x' put $y=0$

$$x^2 = 4$$

$$x = \pm 2$$

x varies from -2 to 2.

$$\therefore \iiint_S F \cdot \vec{n} \, ds = \int_{-2}^2 \int_{-\sqrt{u-x^2}}^{\sqrt{u-x^2}} \int_0^3 (u+4+2z) \, dx \, dy \, dz$$

$$= \int_{-2}^2 \int_0^3 \left[\int_{-\sqrt{u-x^2}}^{\sqrt{u-x^2}} (u-4y+2z) \, dy \right] dz \, dx$$

$$= \int_{-2}^2 \int_0^3 \left(4y - \frac{4y^2}{2} + 2zy \right) \Big|_{-\sqrt{u-x^2}}^{\sqrt{u-x^2}} dz \, dx$$

$$= \int_{-2}^2 \int_0^3 \left((u+2z)y - 2y^2 \right) \Big|_{-\sqrt{u-x^2}}^{\sqrt{u-x^2}} dz \, dx$$

$$= \int_{-2}^2 \int_0^3 \left((u+2z)(\sqrt{u-x^2} + \sqrt{u-x^2}) - 2(\sqrt{u-x^2})^2 + 2(\sqrt{u-x^2})^2 \right) dz \, dx$$

$$= \int_{-2}^2 \int_0^3 \left((u+2z)(2\sqrt{u-x^2}) - 0 \right) dz \, dx$$

$$= \int_{-2}^2 \left[2\sqrt{u-x^2} \left(4z + \frac{2z^2}{2} \right) \Big|_0^3 \right] dx$$

$$= \int_{-2}^2 \left[2\sqrt{u-x^2} \left(u(3) + (3)^2 \right) + 16(3) + 12x^2 \right] dx$$

$$= \int_{-2}^2 \left(2\sqrt{u-x^2} (12+9) + 16(3) + 12x^2 \right) dx$$

$$= \int_{-2}^2 \left(42\sqrt{u-x^2} + 48 + 12x^2 \right) dx$$

$$= 42 \int_{-2}^2 \sqrt{u-x^2} \, dx + 48 \int_{-2}^2 dx + 12 \int_{-2}^2 x^2 \, dx$$

$$= 42(2) \int_0^2 \sqrt{2^2-x^2} \, dx + 48 \left[x \right]_{-2}^2 + 12 \left[\frac{x^3}{3} \right]_{-2}^2$$

$$= 84 \left[\frac{x}{2} \sqrt{u-x^2} + \frac{u^2}{2} \sin^{-1} \left(\frac{x}{2} \right) \right]_0^2 + 48(24) + \frac{12}{3} (818)$$

$$= 84 \left[\frac{2}{2} \sqrt{u-4} + 2 \sin^{-1} \left(\frac{2}{2} \right) \right]$$

$$= 84(0) + 168 \sin^{-1}(1)$$

$$= 108 \left(\frac{\pi}{2} \right)$$

$$= 84 \pi //$$

2. verify divergence theorem for $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$ taken over the rectangular parallelepiped $x \geq 0, x \leq a, y \geq 0, y \leq b, z \geq 0, z \leq c$.

Sol:- Given that $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$

To verify divergence theorem, we have to show that

$$\iint_S \vec{F} \cdot \vec{n} ds = \iiint_V \text{div } \vec{F} dv$$

Note:-

* Surface integral can also be written $\iint_S \vec{F} \cdot d\vec{n} ds = \iint_S F_1 dydz + F_2 dzdx + F_3 dxdy$.

1. Apply divergence theorem to evaluate $\iint_S (x+z) dydz + (y+z) dzdx + (x+y) dxdy$. Where 'S' is the surface of the sphere $x^2 + y^2 + z^2 = 4$.

Sol:- Given integral is

$$\iint_S (x+z) dydz + (y+z) dzdx + (x+y) dxdy$$

Here $F_1 = x+z, F_2 = (y+z), F_3 = x+y$.

$$\vec{F} = (x+z)\vec{i} + (y+z)\vec{j} + (x+y)\vec{k}$$

by divergence theorem

$$\iint_S \vec{F} \cdot \vec{n} ds = \iiint_V \text{div } \vec{F} dv$$

$$\text{div } \vec{F} = \frac{\partial}{\partial x}(x+z) + \frac{\partial}{\partial y}(y+z) + \frac{\partial}{\partial z}(x+y)$$

$$= 1 + 1 + 0$$

$$= 2$$

$$\iint_S (x+z) dydz + (y+z) dzdx + (x+y) dxdy = \iiint_V 2 dv = 2 (\text{volume of sphere})$$

$$= 2 \left(\frac{4}{3} \pi r^3 \right)$$

$$= \frac{8}{3} \pi (2)^3$$

$$= \frac{8}{3} \pi (8)$$

$$= \frac{64}{3} \pi //$$

$$\therefore r=2.$$

2. Apply Gauss divergence theorem evaluate $\oint_S x^2 dy dz + x^2 y dz dx + x^2 z dx dy$, where 'S' is the closed surface bounded by the plane $z=0$, $z=b$ and the cylinder $x^2+y^2=a^2$

Sol:- Given integral is

$$\oint_S x^2 dy dz + x^2 y dz dx + x^2 z dx dy$$

$$\text{here } F_1 = x^3, F_2 = x^2 y, F_3 = x^2 z$$

$$\vec{F} = x^3 \vec{i} + x^2 y \vec{j} + x^2 z \vec{k}$$

$$\text{div } \vec{F} = \frac{\partial}{\partial x} x^3 + \frac{\partial}{\partial y} x^2 y + \frac{\partial}{\partial z} x^2 z$$

$$= 3x^2 + x^2 + x^2$$

$$= 5x^2$$

by divergence theorem

$$\oint_S \vec{F} \cdot \vec{n} ds = \iiint_V \text{div } \vec{F} dv$$

$$\oint_S x^2 dy dz + x^2 y dz dx + x^2 z dx dy = \iiint_V 5x^2 dx dy dz \rightarrow (1)$$

where 'V' is the volume bounded by $z=0$, $z=b$
 $x^2+y^2=a^2$

Limits:- z varies from 0 to b

$$x^2+y^2=a^2$$

$$y^2 = a^2 - x^2$$

$$y = \pm \sqrt{a^2 - x^2}$$

y varies from $-\sqrt{a^2 - x^2}$ to $\sqrt{a^2 - x^2}$

$$x^2 = a^2$$

$$x = \pm a$$

x varies from -a to a

from eqn ①

$$I = \int_{x=-a}^a \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{z=0}^b 5x^2 dx dy dz$$

$$= \int_{x=-a}^a \int_0^b 5x^2 \left(\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy \right) dz dx$$

$$= \int_{-a}^a \int_0^b 5x^2 (y) \Big|_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dz dx$$

$$= \int_{-a}^a \int_0^b 5x^2 (\sqrt{a^2-x^2} + \sqrt{a^2-x^2}) dz dx$$

$$= \int_{-a}^a \int_0^b 5x^2 (2\sqrt{a^2-x^2}) dz dx$$

$$= \int_{-a}^a 5x^2 (2\sqrt{a^2-x^2}) \left(\int_0^b dz \right) dx$$

$$= \int_{-a}^a 5x^2 (2\sqrt{a^2-x^2}) [z]_0^b dx$$

$$= \int_{-a}^a 10x^2 \sqrt{a^2-x^2} (b) dx$$

$$= 10b \int_{-a}^a x^2 \sqrt{a^2-x^2} dx$$

$$= 10b(2) \int_0^a x^2 \sqrt{a^2-x^2} dx$$

put $x = a \sin \theta$
 $dx = a \cos \theta d\theta$

U.L $\Rightarrow x = a \Rightarrow a = a \sin \theta$
 $\sin \theta = 1$
 $\theta = \pi/2$

L.L $\Rightarrow x = 0 \Rightarrow 0 = a \sin \theta$
 $\sin \theta = 0$
 $\theta = 0$

$$I = 20b \int_0^{\pi/2} a^2 \sin^2 \theta \sqrt{a^2 - a^2 \sin^2 \theta} (a \cos \theta d\theta)$$

$$= 20ab \int_0^{\pi/2} a^2 \sin^2 \theta a \sqrt{1 - \sin^2 \theta} (a \cos \theta d\theta)$$

$$= 20ab \int_0^{\pi/2} a^4 \sin^2 \theta \cos \theta \sqrt{1 - \sin^2 \theta} d\theta$$

$\therefore \int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$
 where $f(x)$ is even

$$\begin{aligned}
 &= 20ba^4 \int_0^{\pi/2} \sin^2 \theta \cos^2 \theta \cos \theta d\theta \\
 &= 20ba^4 \int_0^{\pi/2} \sin^2 \theta \cos^3 \theta d\theta \\
 &= 20ba^4 \int_0^{\pi/2} \sin^2 \theta (1 - \sin^2 \theta) d\theta \\
 &= 20ba^4 \left[\int_0^{\pi/2} \sin^2 \theta d\theta - \int_0^{\pi/2} \sin^4 \theta d\theta \right] \\
 &= 20ba^4 \left(\frac{1}{2} \times \frac{\pi}{2} - \frac{3 \times 1}{4 \times 2} \times \frac{\pi}{2} \right) \quad \because \int_0^{\pi/2} \sin^m \theta d\theta = \int_0^{\pi/2} \cos^m \theta d\theta \\
 &= 20a^4b \left(\frac{\pi}{4} - \frac{3\pi}{16} \right) = \frac{(m-1)(m-3) \dots \dots 1}{m(m-2) \dots \dots 2} \times \frac{\pi}{2} \text{ if } m \text{ is even} \\
 &= 20a^4b \left(\frac{\pi}{16} \right)
 \end{aligned}$$

$$I = \frac{5}{4} a^4 b \pi //$$

Stoke's theorem :-

1. Evaluate by Stoke's theorem $\oint_C (x+y) dx + (2x-z) dy + (y+z) dz$.
Where 'C' is the boundary of the triangle with vertices (0,0,0), (1,0,0) & (1,1,0)

Sol:- Given intergal is

$$\oint_C (x+y) dx + (2x-z) dy + (y+z) dz$$

$$\text{here } f_1 = x+y, f_2 = 2x-z, f_3 = y+z$$

$$\vec{F} = (x+y)\vec{i} + (2x-z)\vec{j} + (y+z)\vec{k}$$

by Stoke's theorem

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \vec{n} ds$$

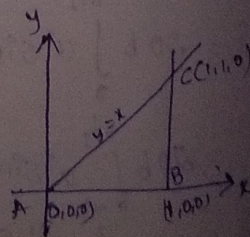
here 'S' is the surface bounded by the curve 'C'.

'C' is a triangle which vertices (0,0,0), (1,0,0) & (1,1,0)

in the above points z quadrate is '0'. that is the triangle lies on xy-plane

limits :- x varies from 0 to 1

y varies from 0 to x



Now $\text{curl } \vec{F} = \nabla \times \vec{F}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-y & y+z \end{vmatrix}$$

$$= \vec{i} \left(\frac{\partial}{\partial y} (y+z) - \frac{\partial}{\partial z} (2x-y) \right) - \vec{j} \left(\frac{\partial}{\partial x} (y+z) - \frac{\partial}{\partial z} (x+y) \right) + \vec{k} \left(\frac{\partial}{\partial x} (2x-y) - \frac{\partial}{\partial y} (x+y) \right)$$

$$= \vec{i} (1+0) - \vec{j} (0-0) + \vec{k} (2-1)$$

$$= 2\vec{i} + \vec{k}$$

then $\text{curl } \vec{F} \cdot \vec{n} = (2\vec{i} + \vec{k}) \cdot \vec{k}$

$$= 1$$

then $ds = dx dy$

$$\oint_C (x+y)dx + (2x-z)dy + (y+z)dz = \iint_S (\text{curl } \vec{F} \cdot \vec{n}) dx dy$$

$$= \int_{x=0}^1 \int_{y=0}^x dx dy$$

$$= \int_0^1 (y)_0^x dx$$

$$= \int_0^1 x dx$$

$$= \left(\frac{x^2}{2} \right)_0^1$$

$$= \frac{1}{2}$$

2. Apply Stoke's theorem to evaluate $\oint_C ydx + zdy + xdz$ where 'C' is the curve of intersection of sphere $x^2 + y^2 + z^2 = a^2$ and the plane $x+z=a$.

Sol:- Given integral

$$\oint_C ydx + zdy + xdz$$

$$\vec{F} = y\vec{i} + z\vec{j} + x\vec{k}$$

by Stoke's theorem

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \vec{n} ds$$

Here 's' is the surface bounded by 'c' and 'c' is the intersection of sphere & plane. is a circular disk of diameter $\overline{AB}$.

$$\text{diameter } AB = \sqrt{(0-a)^2 + (0-0)^2 + (a-0)^2} \\ = \sqrt{a^2 + a^2}$$

$$AB = \sqrt{2} \cdot a$$

$$\text{and radius of the circle} = \frac{1}{2} AB \\ = \frac{1}{2} \sqrt{2} \cdot a \\ = \frac{a}{\sqrt{2}}$$

Here the region lies on the plane $x+z=a$

$$\text{let } \phi = x+z-a$$

$$\nabla \phi = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}$$

$$\nabla \phi = \vec{i} + \vec{k}, \text{ is the normal to the surface}$$

$$\therefore \text{unit normal vector} = \frac{\nabla \phi}{|\nabla \phi|}$$

$$\vec{n} = \frac{\vec{i} + \vec{k}}{\sqrt{2}}$$

$$\text{Now } \text{curl } \vec{F} = \nabla \times \vec{F}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$$

$$= \vec{i} \left(\frac{\partial}{\partial y} x - \frac{\partial}{\partial z} z \right) - \vec{j} \left(\frac{\partial}{\partial x} x - \frac{\partial}{\partial z} y \right) + \vec{k} \left(\frac{\partial}{\partial x} z - \frac{\partial}{\partial y} x \right)$$

$$= \vec{i}(-1) - \vec{j}(1) + \vec{k}(-1)$$

$$= -\vec{i} - \vec{j} - \vec{k}$$

$$\therefore \text{curl } \vec{F} \cdot \vec{n} = (-\vec{i} - \vec{j} - \vec{k}) \cdot \frac{\vec{i} + \vec{k}}{\sqrt{2}}$$

$$= \frac{-1-1}{\sqrt{2}} = -\sqrt{2}$$

$$\therefore \int_C y dx + z dy + x dz = \iint_S -\sqrt{2} dS$$

$$= -\sqrt{2} \iint_S dS$$

$$= -\sqrt{2} (\text{Area of circle})$$

$$= -\sqrt{2} (\pi r^2)$$

$$= -\sqrt{2} \left[\pi \left(\frac{a}{\sqrt{2}} \right)^2 \right]$$

$$= -\sqrt{2} \frac{\pi a^2}{2}$$

$$= \frac{-\pi a^2}{\sqrt{2}} //$$

problems on Green's theorem:-

1. Verify Green's theorem in the plane for $\oint_C (3x^2 - 8y^2) dx + (4y - 6xy) dy$, where C is the region defined by $y = \sqrt{x}$ and $y = x^2$.

Sol:- The given curves be $y = \sqrt{x}$ and $y = x^2$.

Solving these eqn, the points of intersection are

$O(0,0)$ and $A(1,1)$.

To Verify Green's theorem:
to prove that

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

consider

$$\oint_C M dx + N dy = \int_{ODA} (3x^2 - 8y^2) dx + (4y - 6xy) dy + \int_{AEO} (3x^2 - 8y^2) dx + (4y - 6xy) dy \rightarrow (1)$$

Along ODA, $y = x^2$, $dy = 2x dx$ and x varies from 0 to 1.

Along AEO, $x = y^2$, $dx = 2y dy$ and y varies from 1 to 0.

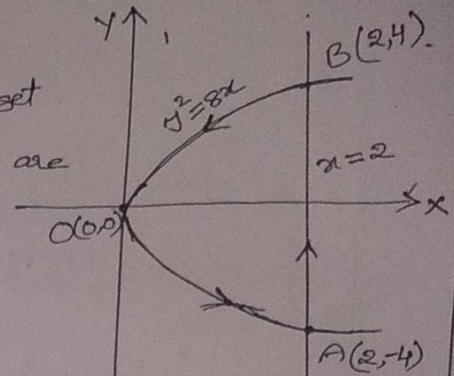
$$\begin{aligned} \int_{ODA} (3x^2 - 8y^2) dx + (4y - 6xy) dy &= \int_0^1 (3x^2 - 8x^4) dx + (4x^2 - 6x^3)(2x dx) \\ &= \int_0^1 (3x^2 + 8x^3 - 20x^4) dx \\ &= \left[3 \frac{x^3}{3} + 8 \frac{x^4}{4} - 20 \frac{x^5}{5} \right]_0^1 \\ &= 1 + 2 - 4 = -1. \end{aligned}$$

2. Evaluate $\oint_C (x^2 - 2xy) dx + (x^2y + 3) dy$ around the boundary of the region defined by $y^2 = 8x$ and $x = 2$.

i) directly ii) By using Green's theorem.

Sol:

By solving $y^2 = 8x$ and $x = 2$, we get the point of intersection A and B are $(2, -4)$ and $(2, 4)$ respectively.



i) Directly:

$$\text{Consider } \oint_C (x^2 - 2xy) dx + (x^2y + 3) dy = \int_{AB} (x^2 - 2xy) dx + (x^2y + 3) dy + \int_{BOA} (x^2 - 2xy) dx + (x^2y + 3) dy \quad \text{--- (1)}$$

Along AB, $x = 2$, $dx = 0$ and 'y' varies from -4 to 4 .

$$\begin{aligned} \therefore \int_{AB} (x^2 - 2xy) dx + (x^2y + 3) dy &= \int_{y=-4}^4 (4y + 3) dy \quad (\because dx = 0) \\ &= \left[4 \frac{y^2}{2} + 3y \right]_{-4}^4 = 24. \end{aligned}$$

$$\begin{aligned} \text{Along BOA, } y^2 = 8x \Rightarrow x &= \frac{y^2}{8}, \quad dx = \frac{2y dy}{8} \\ &\Rightarrow dx = \frac{y dy}{4}. \end{aligned}$$

and also 'y' varies from 4 to -4 .

$$\begin{aligned} \therefore \int_{BOA} (x^2 - 2xy) dx + (x^2y + 3) dy &= \int_{y=4}^{-4} \left(\frac{y^4}{64} - \frac{y^3}{4} \right) \frac{y dy}{4} + \left(\frac{y^4}{64} \cdot y + 3 \right) dy \\ &= \int_4^{-4} \left(\frac{5y^5}{256} - \frac{y^4}{16} + 3 \right) dy \\ &= \left[\frac{5}{256} \cdot \frac{y^6}{6} - \frac{1}{16} \cdot \frac{y^5}{5} + 3y \right]_4^{-4} \\ &= \frac{128}{5} - 24 = \frac{8}{5}. \end{aligned}$$

$\therefore$ from (1),

$$\oint_C (x^2 - 2xy) dx + (x^2y + 3) dy = 24 + \frac{8}{5} = \frac{128}{5}.$$

$$\int_{AEO} (3x^2 - 8y^2) dx + (4y - 6xy) dy = \int_{y=1}^0 (3y^4 - 8y^2)(2y dy) + (4y - 6y^3) dy \quad (41)$$

$$= \int_{y=1}^0 (6y^5 - 22y^3 + 4y) dy = \frac{5}{2}$$

$\therefore$ from (1),

$$\oint_C M dx + N dy = -1 + \frac{5}{2} = \frac{3}{2} \longrightarrow (2)$$

Again consider $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$,

Here $M = 3x^2 - 8y^2$, $N = 4y - 6xy$

$$\frac{\partial M}{\partial y} = -16y \quad \frac{\partial N}{\partial x} = -6y$$

$$\begin{aligned} \therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy &= \int_{x=0}^1 \int_{y=x^2}^{\sqrt{x}} (-6y + 16y) dy dx \\ &= 10 \int_{x=0}^1 \int_{y=x^2}^{\sqrt{x}} y dy dx \\ &= 10 \int_{x=0}^1 \left(\frac{y^2}{2} \right)_{y=x^2}^{\sqrt{x}} dx \\ &= 5 \int_{x=0}^1 (x - x^4) dx = 5 \left(\frac{x^2}{2} - \frac{x^5}{5} \right)_{x=0}^1 \\ &= 5 \left(\frac{1}{2} - \frac{1}{5} \right) \end{aligned}$$

$$\Rightarrow \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy = \frac{3}{2} \longrightarrow (3)$$

from (2) and (3),

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy = \frac{3}{2}$$

Hence Green's theorem is verified.

19) Using Green's theorem:

(43)

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

Here $M = x^2 - 2xy$ and $N = x^2y + 3$

$$\frac{\partial M}{\partial y} = -2x$$

$$\frac{\partial N}{\partial x} = 2xy$$

$$\oint_C M dx + N dy = \iint_R (2xy + 2x) dx dy$$

$$= \int_{x=0}^2 \int_{y=-\sqrt{8x}}^{\sqrt{8x}} (2xy + 2x) dy dx$$

$$= \int_{x=0}^2 \left[2x \cdot \frac{y^2}{2} + 2xy \right]_{-\sqrt{8x}}^{\sqrt{8x}} dx$$

$$= \int_{x=0}^2 [8x^2 + 2x\sqrt{8x} - 8x^2 + 2x\sqrt{8x}] dx$$

$$= \int_{x=0}^2 (4x\sqrt{8x}) dx = 8\sqrt{2} \int_{x=0}^2 x^{3/2} dx$$

$$= 8\sqrt{2} \left[\frac{x^{5/2}}{5/2} \right]_0^2 = \frac{16\sqrt{2}}{5} \left[2^{5/2} - 0 \right]$$

$$= \frac{16\sqrt{2}}{5} \cdot 2^2 \cdot \sqrt{2}$$

$$\oint_C (x^2 - 2xy) dx + (x^2y + 3) dy = \underline{\underline{\frac{128}{5}}}$$

3.

Apply Green's theorem to evaluate

$\int_C (y - \sin x) dx + \cos x dy$, where C is the plane-triangle enclosed by the lines $y=0$, $x=\pi/2$ and $y=\frac{2}{\pi}x$.

Sol:-

Given integral is $\int_C (y - \sin x) dx + \cos x dy$.

where $M = y - \sin x$ and $N = \cos x$.

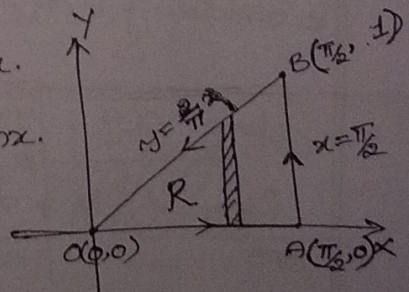
$$\frac{\partial M}{\partial y} = 1$$

$$\frac{\partial N}{\partial x} = -\sin x$$

The given lines are:

$$y=0, x=\pi/2, y=\frac{2}{\pi}x.$$

The co-ordinates of O and A are $(0,0)$ and $(\pi/2, 0)$ respectively.



∴ from ①,

$$\begin{aligned}\oint_C (y - \sin x) dx + \cos x dy &= \iint_R (-\sin x - 1) dx dy \\ &= \int_{x=0}^{\pi/2} \int_{y=0}^{\frac{2}{\pi}x} (-\sin x - 1) dy dx \\ &= \int_{x=0}^{\pi/2} (-\sin x - 1) \left[y \right]_0^{\frac{2}{\pi}x} dx \\ &= -\frac{2}{\pi} \int_{x=0}^{\pi/2} (\sin x + 1) x dx \\ &= -\frac{2}{\pi} \left[x(-\cos x + x) - 1 \cdot (-\sin x + \frac{x^2}{2}) \right]_0^{\pi/2} \\ &= -\frac{2}{\pi} \left[-x \cos x + \frac{x^2}{2} + \sin x \right]_0^{\pi/2} \\ &= -\frac{2}{\pi} \left[0 + \frac{\pi^2}{8} + 1 + 0 - 0 - 0 \right]\end{aligned}$$

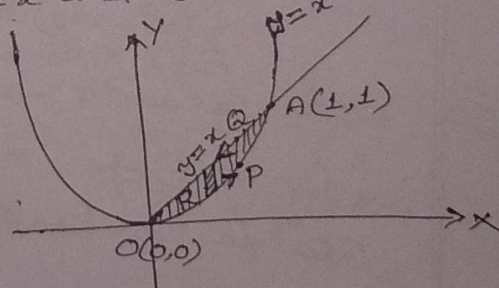
$$\oint_C (y - \sin x) dx + \cos x dy = -\left(\frac{\pi}{4} + \frac{2}{\pi}\right)$$

4. Verify Green's theorem for $\oint_C (xy + y^2) dx + x^2 dy$, where 'C' is bounded by $y=x$ and $y=x^2$.

Sol:-

To prove that:

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$



$$\text{Consider } \oint_C M dx + N dy = \int_{OPA} (xy + y^2) dx + x^2 dy + \int_{AOO} (xy + y^2) dx + x^2 dy$$

Along OPA, $y=x^2$, $dy=2x dx$ and $x \rightarrow 0$ to 1 .

$$\begin{aligned}\int_{OPA} (xy + y^2) dx + x^2 dy &= \int_{x=0}^1 (x^3 + x^4) dx + x^2 (2x dx) \\ &= \int_0^1 (3x^3 + x^4) dx = \frac{19}{20}\end{aligned}$$

Along ABO, $y=x$, $dy=dx$ and $x \rightarrow 1$ to 0 . (45)

$$\int_{ABO} (xy + y^2) dx + x^2 dy = \int_{x=1}^0 (x^2 + x^2) dx + x^2 dx$$

$$= \int_1^0 3x^2 dx = [x^3]_1^0 = -1.$$

$$\therefore \oint_C (xy + y^2) dx + x^2 dy = \frac{17}{20} - 1 = -\frac{1}{20} \rightarrow (1)$$

Also $\iint_R \left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y} \right) dx dy = \int_{x=0}^1 \int_{y=x^2}^x [2x - (x + 2y)] dy dx$

$$= \int_{x=0}^1 [xy - y^2]_{y=x^2}^x dx$$

$$= \int_{x=0}^1 (x^2 - x^2 - x^3 + x^4) dx$$

$$= \left[-\frac{x^3}{3} + \frac{x^5}{5} \right]_0^1 = -\frac{1}{3} + \frac{1}{5} = -\frac{1}{15} \rightarrow (2)$$

from (1) & (2),

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y} \right) dx dy.$$

Hence Green's theorem verified.

5.

Evaluate by Green's theorem, $\oint_C e^{-x} \sin y dx + e^{-x} \cos y dy$, where C is the rectangle whose vertices are $(0,0)$, $(\pi,0)$, $(\pi, \pi/2)$ and $(0, \pi/2)$.

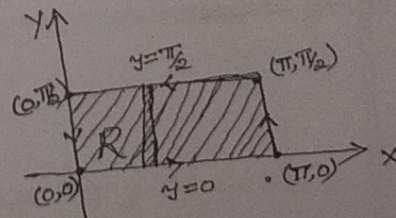
Sol:-

From given integral, we have

$$M = e^{-x} \sin y \quad \text{and} \quad N = e^{-x} \cos y$$

$$\frac{\partial M}{\partial y} = e^{-x} \cos y$$

$$\frac{\partial N}{\partial x} = -e^{-x} \cos y$$



∴ By Green's theorem, we have

$$\begin{aligned} \oint \bar{e}^x \sin y \, dx + \bar{e}^x \cos y \, dy &= \iint_R (-\bar{e}^x \cos y - \bar{e}^x \cos y) \, dx \, dy \\ &= \int_{x=0}^{\pi} \int_{y=0}^{\pi/2} -2\bar{e}^x \cos y \, dx \, dy \\ &= -2 \int_{x=0}^{\pi} \bar{e}^x \left[\sin y \right]_{y=0}^{\pi/2} \, dx \\ &= -2 \int_{x=0}^{\pi} \bar{e}^x \, dx \\ &= -2 \left[\frac{\bar{e}^x}{-1} \right]_{x=0}^{\pi} = 2(\bar{e}^{\pi} - 1). \end{aligned}$$

6.

Verify Green's theorem for $\oint (3x^2 + 2y) \, dx - (x + 3\cos y) \, dy$ around the parallelogram having vertices as $(0,0)$, $(2,0)$, $(3,1)$ and $(1,1)$.

Sol:- To prove that:

$$\oint_C M \, dx + N \, dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dx \, dy.$$

Now $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dx \, dy$

$$= \int_{y=0}^1 \left[\int_{x=y}^{2+y} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dx \right] \, dy.$$

$$= -6 \rightarrow \textcircled{1}$$

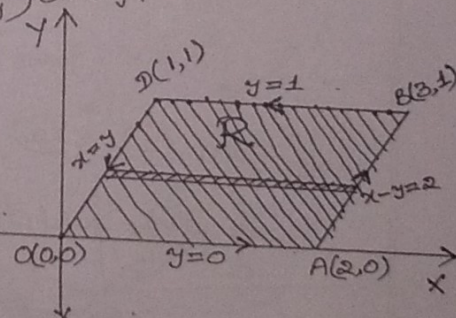
Also $\oint_C M \, dx + N \, dy = \int_{OA} (M \, dx + N \, dy) + \int_{AB} (M \, dx + N \, dy) + \int_{BD} (M \, dx + N \, dy) +$

$$\int_{DO} (M \, dx + N \, dy)$$

$$= -6 \rightarrow \textcircled{2}$$

from $\textcircled{1}$ & $\textcircled{2}$,

Green's theorem is verified.



problems on Gauss divergence theorem:-

(47)

1. Verity divergence theorem for

$\vec{F} = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}$ taken over the rectangular parallelepiped $x \geq 0, x \leq a, y \geq 0, y \leq b, z \geq 0, z \leq c$.

Sol: Given that $\vec{F} = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}$.

To verify divergence theorem, we have to show that

$$\iint_S \vec{F} \cdot \hat{n} \, ds = \iiint_V \text{div} \vec{F} \, dv.$$

First we have to calculate $\iint_S \vec{F} \cdot \hat{n} \, ds$ over the six faces of the rectangular parallelepiped.

i) over the face DEFA, $\hat{n} = \hat{i}, x = a$.

$$\vec{F} \cdot \hat{n} = x^2 - yz.$$

$$\Rightarrow \vec{F} \cdot \hat{n} = a^2 - yz \quad (\because x = a)$$

$$\iint_{DEFA} \vec{F} \cdot \hat{n} \, ds = \int_{z=0}^c \int_{y=0}^b (a^2 - yz) \, dy \, dz$$

$$= \int_{z=0}^c \left[a^2 y - \frac{y^2 z}{2} \right]_{y=0}^b \, dz$$

$$= \int_{z=0}^c \left(a^2 b - \frac{b^2 z}{2} \right) \, dz = \left[a^2 b z - \frac{b^2 z^2}{4} \right]_{z=0}^c$$

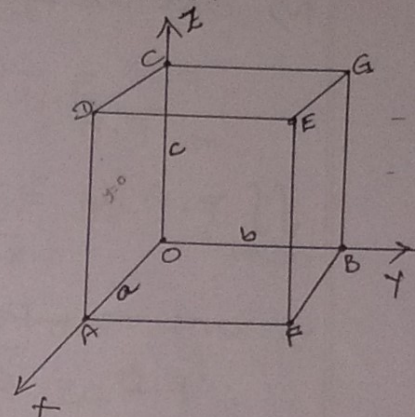
$$\iint_{DEFA} \vec{F} \cdot \hat{n} \, ds = a^2 bc - \frac{b^2 c^2}{4}$$

ii) over the face BGCO, $\hat{n} = -\hat{i}, x = 0$.

$$\vec{F} \cdot \hat{n} = -(x^2 - yz) = yz \quad (\because x = 0)$$

$$\iint_{BGCO} \vec{F} \cdot \hat{n} \, ds = \int_{z=0}^c \int_{y=0}^b yz \, dy \, dz$$

$$= \int_{z=0}^c z \left[\frac{y^2}{2} \right]_{y=0}^b \, dz = \frac{b^2}{2} \left[\frac{z^2}{2} \right]_{z=0}^c = \frac{b^2 c^2}{4}$$



$$\therefore \iint_{BSCO} \vec{F} \cdot \hat{n} \, ds = \frac{b^2 c^2}{4}$$

iii) over the face BGEF, $\hat{n} = \hat{j}$, $y=b$.

$$\vec{F} \cdot \hat{n} = y^2 - zx = b^2 - zx \quad (\because y=b)$$

$$\iint_{BGEF} \vec{F} \cdot \hat{n} \, ds = \int_{z=0}^c \int_{x=0}^a (b^2 - zx) \, dx \, dz$$

$$\Rightarrow \iint_{BGEF} \vec{F} \cdot \hat{n} \, ds = abc - \frac{a^2 c^2}{4}$$

iv) over the face OCDA, $\hat{n} = -\hat{j}$, $y=0$

$$\vec{F} \cdot \hat{n} = -(y^2 - zx)$$

$$\Rightarrow \vec{F} \cdot \hat{n} = zx \quad (\because y=0)$$

$$\iint_{OCDA} \vec{F} \cdot \hat{n} \, ds = \int_{z=0}^c \int_{x=0}^a zx \, dx \, dz = \frac{a^2 c^2}{4}$$

v) over the face CDEG, $\hat{n} = \hat{k}$, $z=c$

$$\therefore \vec{F} \cdot \hat{n} = z^2 - xy = c^2 - xy \quad (\because z=c)$$

$$\iint_{CDEG} \vec{F} \cdot \hat{n} \, ds = \int_{y=0}^b \int_{x=0}^a (c^2 - xy) \, dx \, dy$$

$$\Rightarrow \iint_{CDEG} \vec{F} \cdot \hat{n} \, ds = c^2 ab - \frac{a^2 b^2}{4}$$

vi) over the face AFBO, $\hat{n} = -\hat{k}$, $z=0$.

$$\vec{F} \cdot \hat{n} = -(z^2 - xy) = xy \quad (\because z=0)$$

$$\iint_{AFBO} \vec{F} \cdot \hat{n} \, ds = \int_{y=0}^b \int_{x=0}^a xy \, dx \, dy = \frac{a^2 b^2}{4}$$

Adding the integrals over the six faces,

$$\iint_S \vec{F} \cdot \hat{n} \, ds = a^2 bc + abc^2 + abc^2$$

$$\Rightarrow \iint_S \vec{F} \cdot \hat{n} \, ds = abc(a+b+c) \longrightarrow \textcircled{1}$$

Now $\text{div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy)$ (49)
 $\Rightarrow \text{div } \vec{F} = 2x + 2y + 2z$

$$\begin{aligned} \iiint_V \text{div } \vec{F} \, dv &= \iiint_V 2(x + y + z) \, dv \\ &= 2 \int_{z=0}^c \int_{y=0}^b \int_{x=0}^a (x + y + z) \, dx \, dy \, dz \\ &= 2 \int_{z=0}^c \int_{y=0}^b \left[\frac{x^2}{2} + xy + xz \right]_{x=0}^a \, dy \, dz \\ &= 2 \int_{z=0}^c \int_{y=0}^b \left(\frac{a^2}{2} + ay + az \right) \, dy \, dz \\ &= 2 \int_{z=0}^c \left[\frac{a^2}{2} y + a \frac{y^2}{2} + az y \right]_{y=0}^b \, dz \\ &= 2 \int_{z=0}^c \left[\frac{a^2 b}{2} + \frac{ab^2}{2} + abz \right] \, dz \\ &= 2 \left[\frac{a^2 b}{2} z + \frac{ab^2}{2} z + ab \frac{z^2}{2} \right]_{z=0}^c \\ &= 2 \left[\frac{a^2 bc}{2} + \frac{ab^2 c}{2} + \frac{abc^2}{2} \right] \end{aligned}$$

$$\iiint_V \text{div } \vec{F} \, dv = abc(a + b + c) \longrightarrow (2)$$

From (1) and (2),

$$\iint_S \vec{F} \cdot \hat{n} \, ds = \iiint_V \text{div } \vec{F} \, dv.$$

Hence, Gauss divergence theorem is verified.

2. Evaluate $\iint_S \vec{F} \cdot \hat{n} \, ds$, where $\vec{F} = 2x^2y \hat{i} - y^2 \hat{j} + 4xz^2 \hat{k}$
 and 'S' is the surface $y^2 + z^2 = 9$, $x=2$ in the first octant.

Sol. Given that $\vec{F} = 2x^2y \hat{i} - y^2 \hat{j} + 4xz^2 \hat{k}$

$$\text{div } \vec{F} = 4xy - 2y + 8xz.$$

By Gauss divergence theorem,

$$\iint_S \vec{F} \cdot \hat{n} \, ds = \iiint_V \text{div } \vec{F} \, dv$$

$$\Rightarrow \iiint_S \vec{F} \cdot \hat{n} \, ds = \iiint_V (4xy - 2y + 8xz) \, dx \, dy \, dz$$

$$= \int_{x=0}^2 \int_{y=0}^3 \int_{z=0}^{\sqrt{9-y^2}} (4xy - 2y + 8xz) \, dz \, dy \, dx$$

$$= \int_{x=0}^2 \int_{y=0}^3 \left[4xyz - 2yz + 8x \cdot \frac{z^2}{2} \right]_{z=0}^{\sqrt{9-y^2}} dy \, dx$$

$$= \int_{x=0}^2 \int_{y=0}^3 \left[4xy\sqrt{9-y^2} - 2y\sqrt{9-y^2} + 4x(9-y^2) \right] dy \, dx$$

$$= \int_{y=0}^3 \left[4y\sqrt{9-y^2} \cdot \frac{x^2}{2} - 2y\sqrt{9-y^2} x + 4(9-y^2) \frac{x^2}{2} \right]_{x=0}^2 dy$$

$$= \int_{y=0}^3 \left[8y\sqrt{9-y^2} - 4y\sqrt{9-y^2} + 8(9-y^2) \right] dy$$

$$= \int_{y=0}^3 \left[4y\sqrt{9-y^2} + 8(9-y^2) \right] dy \Rightarrow \int_0^3 y\sqrt{9-y^2} dy + 72 \int_0^3 dy - 8 \int_0^3 y^2 dy$$

$$= -2 \int_0^3 (9-y^2)^{3/2} dy + 72(y)_0^3 - 8 \left(\frac{y^3}{3} \right)_0^3$$

$$= -2 \left[\frac{(9-y^2)^{3/2}}{3/2} \right]_0^3 + 72(3) - \frac{8}{3}(27)$$

$$= \int_{\theta=0}^{\pi/2} \left[12 \sin \theta \sqrt{9-9\sin^2 \theta} + 8(9-9\sin^2 \theta) \right] (3 \cos \theta d\theta)$$

$$\theta=0 \Rightarrow -\frac{4}{3}(9-9\sin^2 \theta)^{3/2} + 216 - 72$$

$$= \int_{\theta=0}^{\pi/2} \left[36 \sin \theta \cos \theta + 72 \cos^3 \theta \right] (3 \cos \theta d\theta)$$

$$= \frac{36}{3} (3^2)^{3/2} + 144 \Rightarrow \frac{36}{3}(27) + 144 \Rightarrow 36 + 144 \Rightarrow 180$$

$$= 108 \cdot \frac{\left[\frac{1+1}{2} \right] \cdot \left[\frac{2+1}{2} \right]}{2 \sqrt{\frac{1+2+2}{2}}} + \frac{72}{216} \cdot \frac{2}{3} \cdot 1$$

$$= 54 \cdot \frac{1 \cdot \sqrt{3/2}}{\sqrt{5/2}} + 144$$

$$= 54 \cdot \frac{\frac{1}{2} \sqrt{1/2}}{\frac{3}{2} \sqrt{1/2}} + 144 = 36 + 144 = 180$$

$$(\because \int_0^{\pi/2} \cos^p \theta d\theta; \text{ where } p \text{ is odd})$$

$$= \frac{p-1}{p} \cdot \frac{p-3}{p-2} \cdots \frac{2}{3} \cdot 1$$

$$= \frac{(p-1)}{p} \cdot \frac{(p-3)}{p-2} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

if p is even

3.

Use divergence theorem, to evaluate $\iiint_S \vec{F} \cdot d\vec{s}$; (5)

Where $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ and S is the surface bounded by the region $x^2 + y^2 = 4$, $z=0$ and $z=3$.

Sol:- By Gauss divergence theorem,

$$\iiint_S \vec{F} \cdot \hat{n} ds = \iiint_V \vec{F} \cdot d\vec{s} = \iiint_V \text{div} \vec{F} dv$$

$$\Rightarrow \iiint_S \vec{F} \cdot \hat{n} ds = \iiint_V (4 - 4y + 2z) dv$$

$$= \int_{x=-2}^2 \int_{z=0}^3 \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - 4y + 2z) dy dz dx.$$

$$= \int_{x=-2}^2 \int_{z=0}^3 \left[4y - 4 \cdot \frac{y^2}{2} + 2zy \right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dz dx$$

$$= \int_{x=-2}^2 \int_{z=0}^3 \left[4\sqrt{4-x^2} - 2(4-x^2) + 2z\sqrt{4-x^2} + 4\sqrt{4-x^2} + 2(4-x^2) + 2z\sqrt{4-x^2} \right] dz dx.$$

$$= \int_{x=-2}^2 \int_{z=0}^3 \left[8\sqrt{4-x^2} + 4z\sqrt{4-x^2} \right] dz dx$$

$$= \int_{x=-2}^2 \left[8\sqrt{4-x^2} z + 4\sqrt{4-x^2} \cdot \frac{z^2}{2} \right]_{z=0}^3 dx$$

$$= \int_{x=-2}^2 \left[24\sqrt{4-x^2} + 18\sqrt{4-x^2} \right] dx$$

$$= \int_{x=-2}^2 42\sqrt{4-x^2} dx = 42 \times 2 \int_{x=0}^2 \sqrt{4-x^2} dx \quad (\because \sqrt{4-x^2} \text{ is an even function})$$

$$= 84 \left[\frac{x}{2} \sqrt{4-x^2} + \frac{4}{2} \sin^{-1}\left(\frac{x}{2}\right) \right]_{x=0}^2$$

$$= 84 [0 + 2 \cdot \sin^{-1}(1) - 0 - 0]$$

$$= 84 \left[2 \cdot \frac{\pi}{2} \right] = \underline{\underline{84\pi}}$$

4.

Verify divergence theorem for $\vec{F} = y\hat{i} + x\hat{j} + z^2\hat{k}$ taken over the cylindrical region bounded by $x^2 + y^2 = 9$, $z=0$ and $z=2$. (52)

Sol: To prove that:

$$\iint_S \vec{F} \cdot \hat{n} \, ds = \iiint_V \text{div} \vec{F} \, dv.$$

Now $\text{div} \vec{F} = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(z^2)$

$$= 0 + 0 + 2z = 2z.$$

$$\iiint_V \text{div} \vec{F} \, dv = \int_{x=-3}^3 \int_{z=0}^2 \int_{y=-\sqrt{9-x^2}}^{\sqrt{9-x^2}} 2z \, dy \, dz \, dx.$$

$$= 2 \int_{x=-3}^3 \int_{z=0}^2 z \cdot \left[y \right]_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dz \, dx$$

$$= 2 \int_{x=-3}^3 \int_{z=0}^2 z (\sqrt{9-x^2} + \sqrt{9-x^2}) dz \, dx.$$

$$= 2 \int_{x=-3}^3 \int_{z=0}^2 z (2\sqrt{9-x^2}) dz \, dx$$

$$= 4 \int_{x=-3}^3 \sqrt{9-x^2} \left[\frac{z^2}{2} \right]_{z=0}^2 dx$$

$$= 2 \int_{x=-3}^3 \sqrt{9-x^2} (4-0) dx$$

$$= 8 \int_{x=-3}^3 \sqrt{9-x^2} dx = 8 \times 2 \int_{x=0}^3 \sqrt{9-x^2} dx$$

$$= 16 \left[\frac{x}{2} \sqrt{9-x^2} + \frac{9}{2} \sin^{-1}\left(\frac{x}{3}\right) \right]_{x=0}^3 \quad \left(\because \sqrt{9-x^2} \text{ is an even fun} \right)$$

$$= 16 \left[0 + \frac{9}{2} \sin^{-1}\left(\frac{3}{3}\right) - 0 - 0 \right]$$

$$= 16 \cdot \left[\frac{9}{2} \cdot \frac{\pi}{2} \right] = 36\pi$$

$$\Rightarrow \iiint_V \text{div} \vec{F} \, dv = 36\pi \longrightarrow \textcircled{1}$$

To evaluate $\iint_S \vec{F} \cdot \hat{n} ds$:-

$$\iint_S \vec{F} \cdot \hat{n} ds = \iint_{S_1} \vec{F} \cdot \hat{n} ds + \iint_{S_2} \vec{F} \cdot \hat{n} ds + \iint_{S_3} \vec{F} \cdot \hat{n} ds$$

for the surface S_1 ,

$$\hat{n} = -\hat{k}, z=0 \text{ and } ds = dxdy.$$

$$\vec{F} \cdot \hat{n} = -z^2 = 0 \quad (\because z=0)$$

$$\iint_{S_1} \vec{F} \cdot \hat{n} ds = \iint_R 0 dxdy = 0.$$

for the surface S_2 , $\hat{n} = \hat{k}$, $z=2$ and $ds = dxdy$.

$$\vec{F} \cdot \hat{n} = z^2 = 4 \quad (\because z=2)$$

$$\iint_{S_2} \vec{F} \cdot \hat{n} ds = \int_{x=-3}^3 \int_{y=-\sqrt{9-x^2}}^{\sqrt{9-x^2}} 4 dy dx$$

$$= 4 \int_{x=-3}^3 \left[y \right]_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dx = 8 \int_{x=-3}^3 \sqrt{9-x^2} dx$$

$$= 16 \int_{x=0}^3 \sqrt{9-x^2} dx = 16 \left[\frac{x}{2} \sqrt{9-x^2} + \frac{9}{2} \sin^{-1} \left(\frac{x}{3} \right) \right]_{x=0}^3$$

$$= 16 \left[0 + \frac{9}{2} \cdot \frac{\pi}{2} - 0 - 0 \right]$$

$$\iint_{S_2} \vec{F} \cdot \hat{n} ds = 36\pi$$

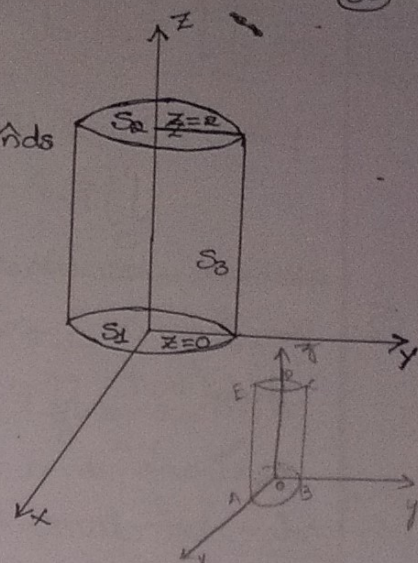
for the surface S_3 , $\hat{n} = \frac{x\hat{i} + y\hat{j}}{3}$ ($\because$ surface S_3 is $\phi = x^2 + y^2 = 9$)

$$\vec{F} \cdot \hat{n} = \frac{xy}{3} + \frac{xy}{3} = \frac{2xy}{3}$$

$$ds = \frac{dxdz}{|\hat{n} \cdot \hat{j}|} = \frac{dxdz}{|y/3|} = \frac{dxdz}{y/3}$$

$$\iint_{S_3} \vec{F} \cdot \hat{n} ds = \iint_R \frac{2xy}{3} \cdot \frac{dxdz}{y/3} = \iint_R 2x dxdz = \int_{z=0}^2 \int_{x=-3}^3 2x dxdz$$

$$\Rightarrow \iint_{S_3} \vec{F} \cdot \hat{n} ds = 2 \int_{z=0}^2 (0) dz = 0 \quad (\because x \text{ is an odd function so } \int_{-3}^3 x dx = 0)$$



$$\therefore \iint_S \vec{F} \cdot \hat{n} \, ds = 0 + 36\pi + 0 = 36\pi. \longrightarrow (2)$$

From (1) and (2),

$$\iint_S \vec{F} \cdot \hat{n} \, ds = \iiint_V \text{div } \vec{F} \, dv$$

Hence, Gauss divergence theorem is verified.

5. If $\vec{F} = ax\hat{i} + by\hat{j} + cz\hat{k}$, a, b, c are constants, show that $\iint_S \vec{F} \cdot \hat{n} \, ds = \frac{4}{3}\pi(a+b+c)$; where 'S' is the surface of a unit sphere. ($x^2 + y^2 + z^2 = 1$)

Sol: By divergence theorem, we have

$$\iint_S \vec{F} \cdot \hat{n} \, ds = \iiint_V \text{div } \vec{F} \, dv.$$

where 'V' is the volume enclosed by 'S'.

$$\text{now } \text{div } \vec{F} = a + b + c.$$

$$\therefore \iint_S \vec{F} \cdot \hat{n} \, ds = \iiint_V (a+b+c) \, dv$$

$$= (a+b+c) \iiint_V dv$$

$$= (a+b+c) \frac{4}{3}\pi. \quad (\because \text{volume of sphere is } \frac{4}{3}\pi r^3 \text{ and } r=1).$$

$$\therefore \iint_S \vec{F} \cdot \hat{n} \, ds = \underline{\underline{(a+b+c) \frac{4}{3}\pi.}}$$

$$\oint_C \vec{F} \cdot d\vec{r} = -4ab^2 \longrightarrow (1)$$

$$\text{Also } \text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(-2y-2y)$$

$$= -4y \hat{k}.$$

The outward unit normal vector perpendicular to xy-plane $\hat{n} = \hat{k}$. and $ds = dxdy$.

$$\therefore \text{curl } \vec{F} \cdot \hat{n} = -4y.$$

$$\iint_S \text{curl } \vec{F} \cdot \hat{n} \, ds = \int_{x=-a}^a \int_{y=0}^b -4y \, dy \, dx$$

$$= \int_{x=-a}^a -4 \left(\frac{y^2}{2} \right)_0^b \, dx = -2 \int_{x=-a}^a b^2 \, dx$$

$$= -2b^2 \left[x \right]_{-a}^a = -2b^2(a+a)$$

$$\iint_S \text{curl } \vec{F} \cdot \hat{n} \, ds = -4ab^2 \longrightarrow (2)$$

From (1) and (2),

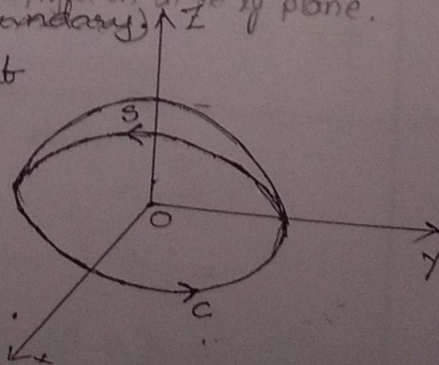
$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \hat{n} \, ds.$$

Hence Stokes's theorem is verified.

2.

Verify Stokes's theorem for $\vec{f} = (2x-y)\hat{i} - yz^2\hat{j} - yz^2\hat{k}$ (where S is the upper half surface of the sphere $x^2+y^2+z^2=1$ and C is its boundary, bounded by its projection on the xy plane).

Sol The projection of the upper half of the sphere $x^2+y^2+z^2=1$ in the xy-plane is the circle $x^2+y^2=1$. Let C be its boundary.



The parametric equations of the circle are:

$$x = \cos \theta, \quad y = \sin \theta, \quad z = 0, \quad 0 \leq \theta \leq 2\pi.$$

Then $\oint_C \vec{F} \cdot d\vec{r} = \int_C (2x-y)dx - yz^2 dy - y^2 z dz$

$$= \int_C (2x-y)dx \quad (\because z=0 \Rightarrow dz=0)$$

$$= \int_{\theta=0}^{2\pi} (2\cos\theta - \sin\theta) \cdot (-\sin\theta d\theta)$$

$$= - \int_{\theta=0}^{2\pi} (\sin 2\theta - \sin^3 \theta) d\theta$$

$$= - \int_0^{2\pi} \left[\sin 2\theta - \left(\frac{1-\cos 2\theta}{2} \right) \right] d\theta$$

$$= - \left[-\frac{\cos 2\theta}{2} - \frac{\theta}{2} + \frac{\sin 2\theta}{4} \right]_0^{2\pi}$$

$$= - \left[-\frac{1}{2} - \frac{2\pi}{2} + 0 + \frac{1}{2} + 0 - 0 \right]$$

$$\oint_C \vec{F} \cdot d\vec{r} = \pi \longrightarrow \textcircled{1}$$

Now $\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2 z \end{vmatrix}$

$$= \hat{i}(-2yz + 2yz) - \hat{j}(0-0) + \hat{k}(0+1)$$

$$\text{curl } \vec{F} = \hat{k}$$

unit normal vector $\hat{n} = \hat{k}$

$$\text{curl } \vec{F} \cdot \hat{n} = 1.$$

Let 'R' be the projection of 'S' in xy-plane. and

$$ds = \frac{dx dy}{|\hat{n} \cdot \hat{k}|} = dx dy.$$

where 'R' is the circle in xy-plane with centre at (0,0) and radius 1.

$$\therefore \iint_S \text{curl } \vec{F} \cdot \hat{n} ds = \iint_R dx dy$$

$$= \pi. \quad (\because \text{Area of } R = \text{area of circle}$$

$$\Rightarrow \iint_S \text{curl } \vec{F} \cdot \hat{n} ds = \pi \longrightarrow \textcircled{2} \quad \left[\begin{array}{l} \text{taking } x \rightarrow 0 \text{ to } 1 \text{ and } y \rightarrow 0 \text{ to } 1, \\ \text{we get } \iint_R dx dy = \pi \end{array} \right]$$

∴ from ① & ②,

$$\oint_C \vec{F} \cdot d\vec{r} = \iiint_S \text{curl } \vec{F} \cdot \hat{n} \, dS$$

Hence Stokes's theorem verified.

3. Evaluate by Stokes's theorem,

$\int_C \sin z \, dx - \cos x \, dy + \sin y \, dz$, where C is the boundary of the rectangle $x \geq 0, x \leq \pi, y \geq 0, y \leq 1, z = 3$.

Sol: From the given integral, we have

$$\vec{F} = \sin z \, \hat{i} - \cos x \, \hat{j} + \sin y \, \hat{k}$$

Here the surface S is the rectangle,

$$x \rightarrow 0 \text{ to } \pi, y \rightarrow 0 \text{ to } 1, z = 3.$$

$$\text{curl } \vec{F} = \cos y \, \hat{i} + \cos z \, \hat{j} + \sin x \, \hat{k}$$

$$\text{Here } \hat{n} = \hat{k}$$

$$\therefore \text{curl } \vec{F} \cdot \hat{n} = \sin x$$

By Stokes's theorem,

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \hat{n} \, dS$$

$$\Rightarrow \int_C (\sin z \, dx - \cos x \, dy + \sin y \, dz) = \int_{x=0}^{\pi} \int_{y=0}^1 \sin x \, dy \, dx$$

$$= \int_{x=0}^{\pi} \sin x \left[y \right]_0^1 dx$$

$$= \left[-\cos x \right]_{x=0}^{\pi}$$

$$= -[\cos \pi - \cos 0]$$

$$= -[-1 - 1]$$

$$\int_C (\sin z \, dx - \cos x \, dy + \sin y \, dz) = \underline{\underline{2}}$$